Matroids and Integrality Gaps for Hypergraphic Steiner Tree Relaxations

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Joint work with:

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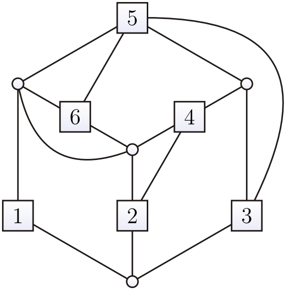
The Steiner Tree problem

Given:
- Graph $G = (V, E)$,
- terminals $R \subseteq V$,
- edge costs $c : E \to \mathbb{R}_+$.

Goal: Find $T \subseteq E$ connecting R & minimizing $c(T) := \sum_{e \in T} c(e)$.



□ : R (terminals)

○ : $V \setminus R$ (Steiner nodes)

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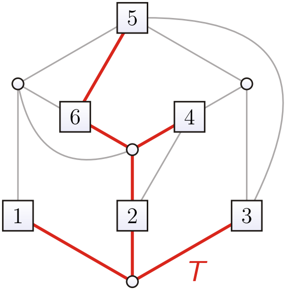

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Approximation results

	Approximation guarantee
Folklore	2
Zelikovsky [1991]	$11/6 \leq 1.83$
Berman & Ramaiyer [1991]	$16/9 \leq 1.78$
Zelikovsky [1993]	$1 + \ln(2) + \epsilon \leq 1.70$
Karpinski & Zelikovsky [1997]	≤ 1.65
Prömel & Steger [2000]	$5/3 \leq 1.67$
Hougardy & Prömel [1999]	≤ 1.59
Robins & Zelikovsky [2005]	$1 + \ln(3)/2 \leq 1.55$
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Major challenge in design of strong approximation algorithms:

➙ Very bad understanding of LP relaxations.

Towards LP-based approaches

Until $(\ln(4) + \epsilon) \approx 1.39$-approx of [Byrka et al., 2010], best Steiner Tree approximations did not use LP relaxations.

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- Relies on a hypergraphic LP relaxation.
- However, only shown that returned solution has cost $\leq 1.39 OPT$, and not $\leq 1.39 LP$. In particular, no implication on integrality gap of LP!
- Algo can be adapted to use $O(\log(n))$ random bits $\Rightarrow$ derandomization.

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➙ We want to get better understanding of hypergraphic relaxation.

Our results

- New algorithm inspired by Byrka et al.:
  - Returns solution that compares against LP.
    $\Rightarrow$ ln(4) bound on integrality gap of LP.
  - Deterministic.
  - Faster: don't need to repeatedly solve LP (as Byrka et al.).

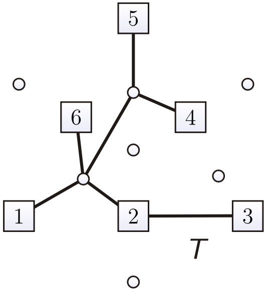
- Deeper understanding of hypergraphic LP.

- Further results for quasi-bipartite graphs.

Hypergraphic LP relaxation

Component relaxation

- Any Steiner Tree can be decomposed into components.
  - A component C is a tree in G whose leaves are terminals and non-leaves are Steiner nodes.
  - For simplicity: denote by C the terminals spanned by component.



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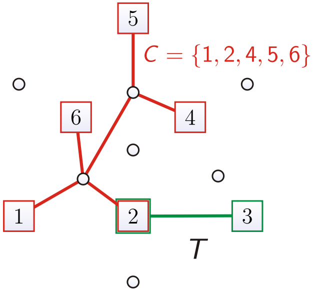

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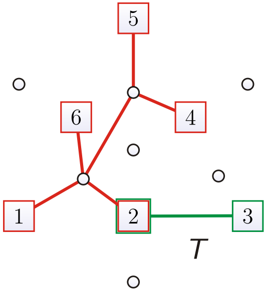

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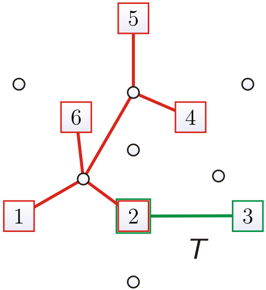

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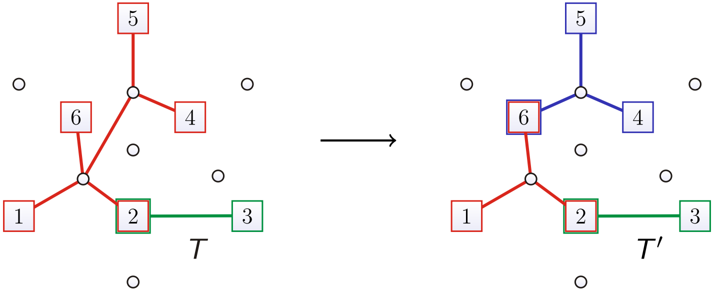
Component relaxation

- Any Steiner Tree can be decomposed into components.
- Goal of hypergraphic LP: determine which components are used.
- Borchers and Du [1997]: sufficient to consider "small" components C: if $|C| \leq 2^r$ is imposed $\Rightarrow$ best such Steiner Trees has cost $\leq 1 + \frac{1}{r}$.



Hypergraphic LP relaxation (component relaxation)

Several equivalent variants of hypergraphic relaxations.
(Warme [1998], Polzin & Vahdati Daneshmand [2003], Könemann et al [2009])

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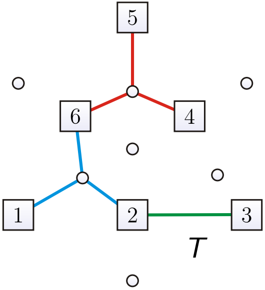

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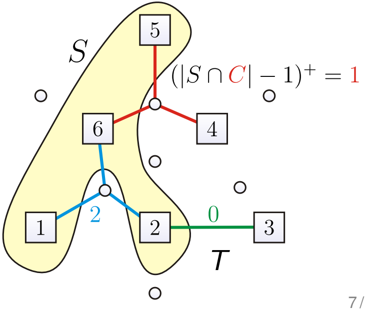

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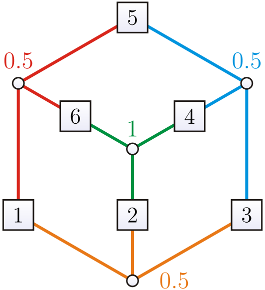

ln(4)-approx of Byrka, Grandoni, Rothvoß & Sanità

(i) Solve hypergraphic LP $\rightarrow (x_C)_{C \in \mathcal{K}}$.

(ii) Contract randomly one component $C \in \mathcal{K}$ with prob $= x_C / (\sum_{C \in \mathcal{K}} x_C)$.

(iii) • If only one terminal left $\rightarrow$ return contracted components.
• Otherwise $\rightarrow$ back to **(i)**.

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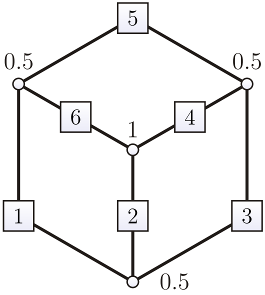

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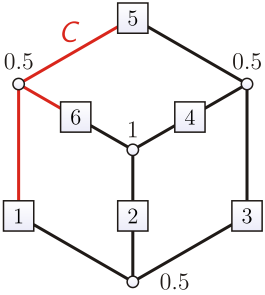

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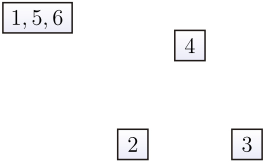

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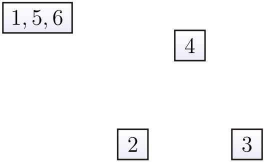

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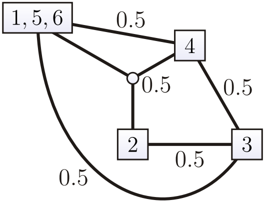

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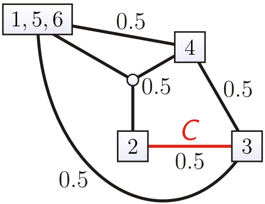

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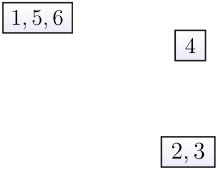

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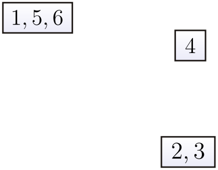

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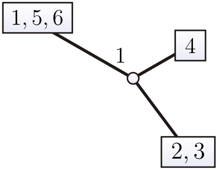

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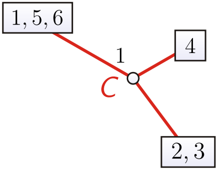

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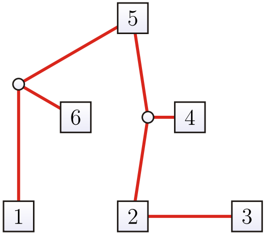


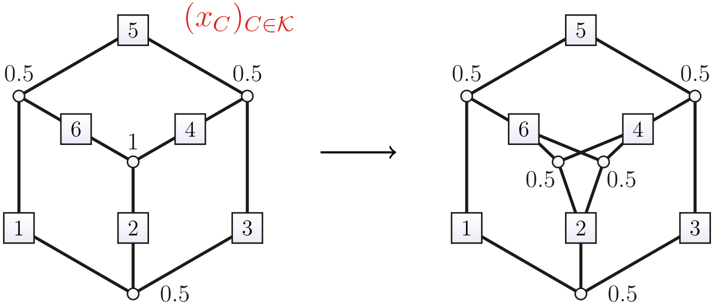
Towards a $ln(4)$ integrality gap

- We want to measure progress in terms of initial LP solution.
- After contraction, we modify current LP solution to obtain feasibility.
  - Modifications: we split up components into smaller ones.

Need better understanding of how components can be split after contraction to obtain feasibility.

Blowup graph

Idea to simplify exposition: assume all components have same LP-value.



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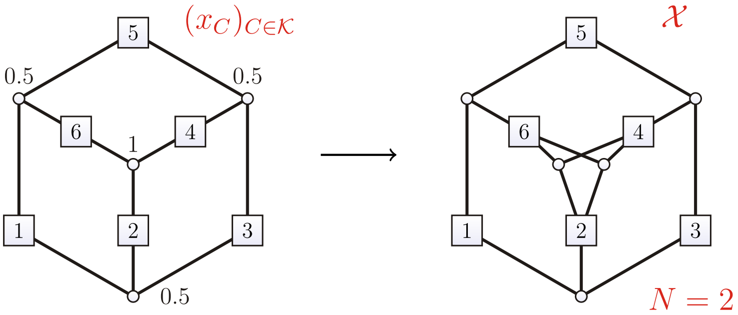

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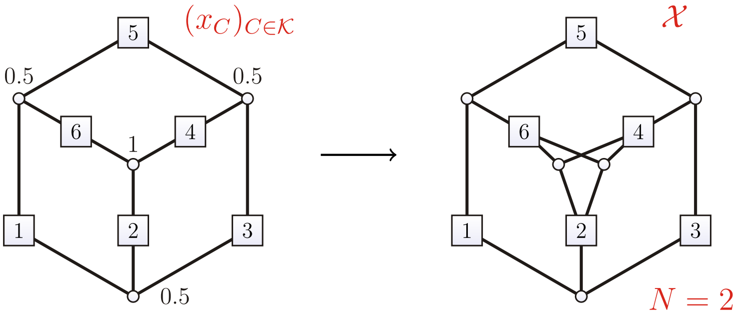

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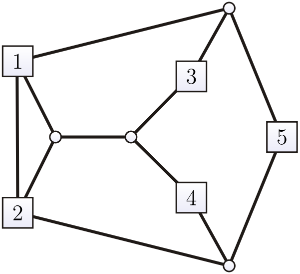
> - $\mathcal{X}$: blowup graph.
> - $\Gamma(\mathcal{X})$: components of blowup graph.
> - $E(\mathcal{X})$: edges of blowup graph.

Edge removals to obtain feasibility



$x_C = 0.5 \; \forall C \in \Gamma(\mathcal{X})$

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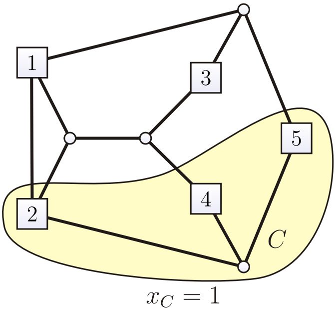

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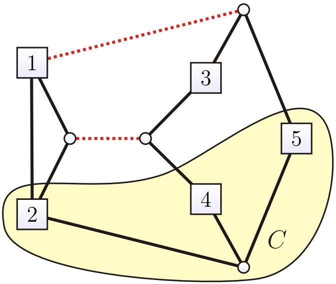

$x_C = 1$

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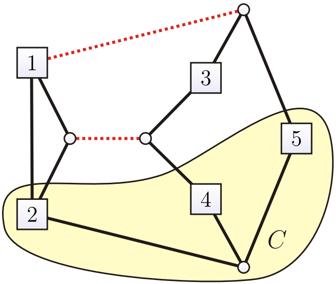

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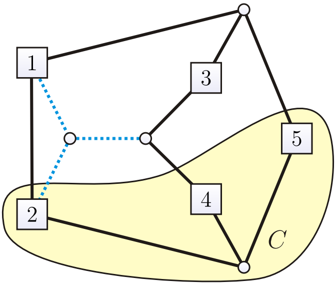

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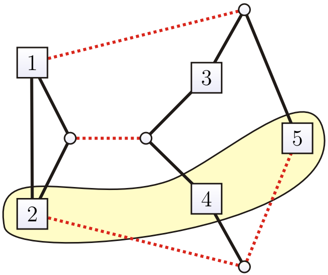

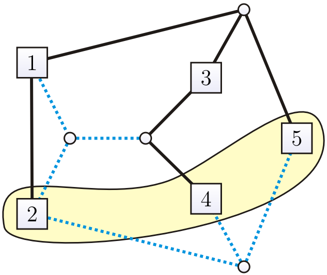

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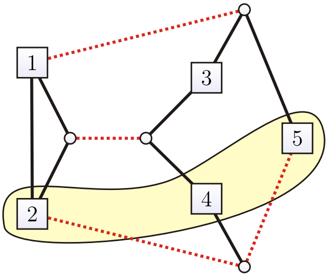

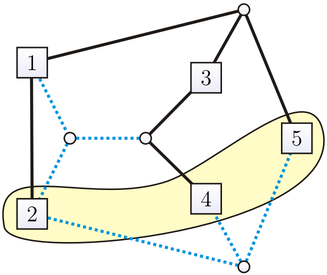

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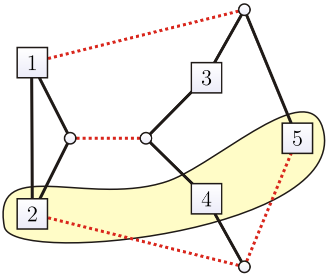

- Feasible sets to remove seem hard to describe directly.
- It turns out that minimal removal sets are well structures:

Theorem

Minimal edge removals achieving feasibility form bases $\mathcal{B}_C$ of a matroid M_C.

Splitting sets and cleanup edges

Possible removals can be described by following two-step procedure:

(i) Remove a basis B of M_C.
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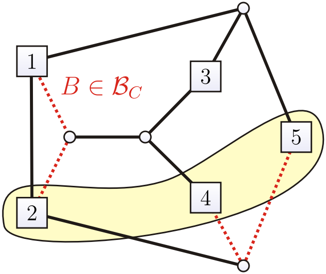

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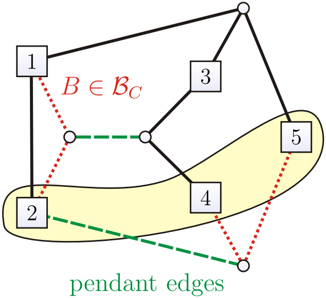

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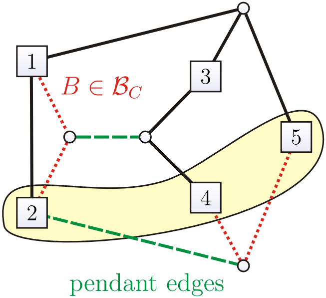

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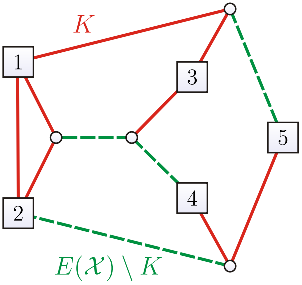

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- Analysis simplifies by partitioning edges at beginning into:
  - Edges to be removed eventually: $\rightarrow$ splitting set: $K \subseteq E(\mathcal{X})$.
  - Edges to be cleaned up eventually: $\rightarrow$ cleanup edges: $E(\mathcal{X}) \setminus K$.

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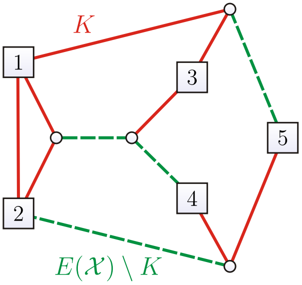

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Splitting set:	Any minimal edge-set whose removal leads to single-terminal connected components.

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- When can an edge $f \in E(\mathcal{X}) \setminus K$ be cleaned up?

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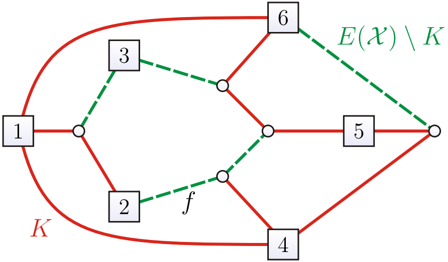

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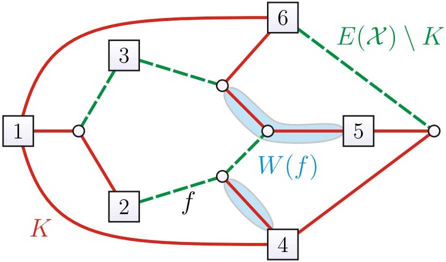

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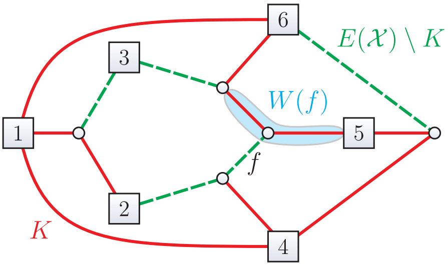

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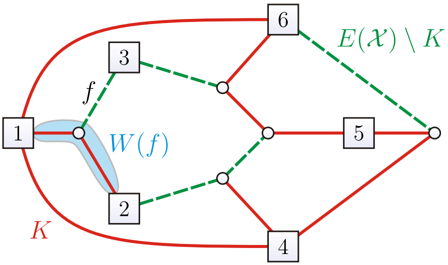

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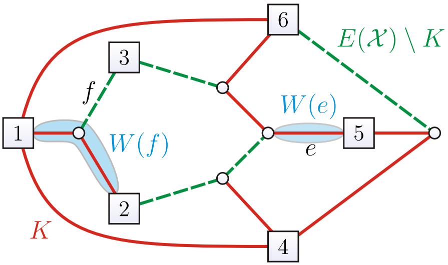
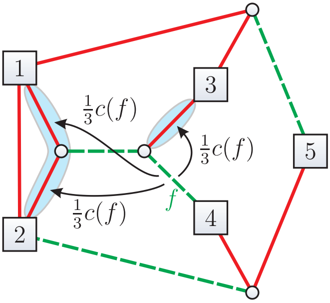


- We set $W(e) = \{e\}$ for $e \in K$.

Measuring progress via a potential function

- We understand quite well how to remove edges from splitting set K.
  - ➙ Measure progress in terms of removed splitting edges.
  - ➙ Plan: charge costs of cleanup edges to splitting edges:

$$w(e) := \frac{1}{N}\left(c(e) + \sum_{\substack{f \in E(\mathcal{X} \setminus K), \\ e \in W(f))}} \frac{c(f)}{|W(f)|} \right).$$



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Potential for blowup graph $\mathcal{X}$ with splitting set K:

$$\Phi_K(\mathcal{X}) := \frac{1}{N} \sum_{e \in E(\mathcal{X})} c(e) H(|W(e)|),$$

- where $H(k) := \sum_{j=1}^{k} \frac{1}{j}$ (harmonic function).

Measuring progress via a potential function

Theorem

(a) When contracting any $C \in \mathcal{K}$, removing $B \subseteq \mathcal{B}_C, B \subseteq K$ and cleaning up: $|\Delta\Phi_K(\mathcal{X})| \geq w(B)$.

(b) $\exists\ C \in \mathcal{K}$ and basis B of M_C, $B \subseteq K$ s.t. $\text{cost}(C) \leq w(B)$.

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$\Rightarrow$ Following algo returns Steiner tree of cost $\leq \Phi_K(\mathcal{X})$:

(i) Solve hypergraphic LP.

(ii) Find $C \in \mathcal{K}$, & basis B of M_C, $B \subseteq K$ s.t. $\text{cost}(C) \leq w(B)$.

(iii) Contract C, remove B & cleanup.

(iv)
- If only one terminal left $\rightarrow$ return contracted components.
- Otherwise $\rightarrow$ back to **(ii)**.

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Some insights how to exploit matroid structure

(b) $\exists\, C \in \mathcal{K}$ and basis B of M_C, $B \subseteq K$ s.t. $\text{cost}(C) \leq w(B)$.

We prove **(b)** via averaging argument by showing:

Lemma

$\exists\, B_C \subseteq K$ basis of M_C $\forall C \in \Gamma(\mathcal{X})$ s.t.:

$$N \cdot w(K) = \sum_{C \in \Gamma(\mathcal{X})} \text{cost}(C) \leq \sum_{C \in \Gamma(\mathcal{X})} w(B_c)$$

High-level proof outline.

(i) We prove $N \cdot \mathbf{1} \in \sum_{C \in \Gamma(\mathcal{X})} P_{M_C}$,

- $\mathbf{1}$ is all-one vector in $\{0,1\}^K$,
- P_{M_C} is matroid polytope of M_C.

(ii) Key insight: $\sum_{C \in \Gamma(\mathcal{X})} P_{M_C}$ is polymatroid with rank funct. $\sum_{C \in \Gamma(\mathcal{X})} r_{M_C}$.

(iii) We prove **(i)** by studying matroids M_C to obtain properties on r_{M_C}.

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Thank you!

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