An $O(1)$-Approximation for Minimum Spanning Tree Interdiction

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What are interdiction problems?

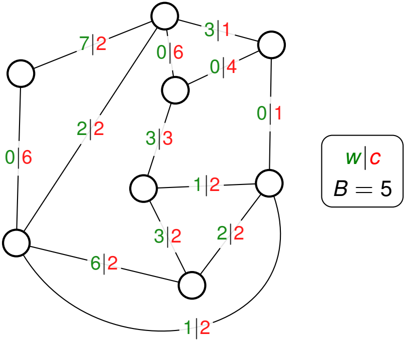
Key question motivating interdiction problems

How sensitive is a system wrt failure/destruction of some of its components?

Interdiction analysis is about worst-case failures.

- Learn how robust a system is.
- Identify weakest spots ⟨ protect system, / attack/inhibit undesirable system/process.

Minimum Spanning Tree (MST) Interdiction



Input

- $G = (V, E)$
- $w : E \to \mathbb{Z}_{\geq 0}$ (weights)
- $c : E \to \mathbb{Z}_{>0}$ (costs)
- $B \in \mathbb{Z}_{>0}$ (budget)

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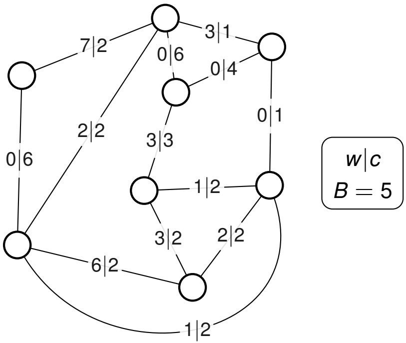

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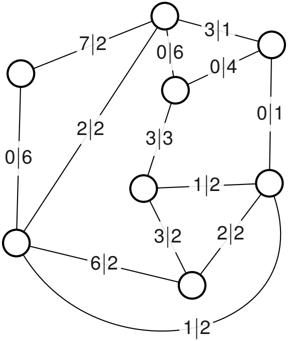

$w|c$
$B = 5$

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Task

Create graph with heavy MST by removing $R \subseteq E$ respecting budget.

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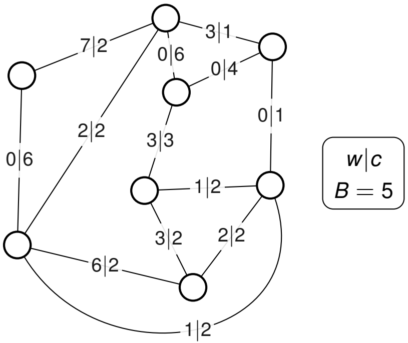

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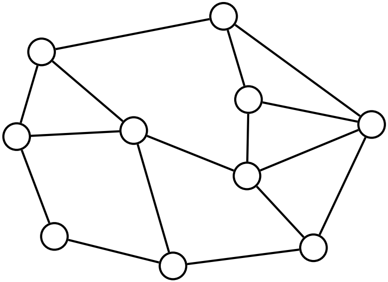
Assumption: Budget not large enough to disconnect graph.

Maximum Component Problem (MCP)

Break graph into max # of connected components by removing q edges.

(k-cut problem where roles of objective and constraint are swapped.)



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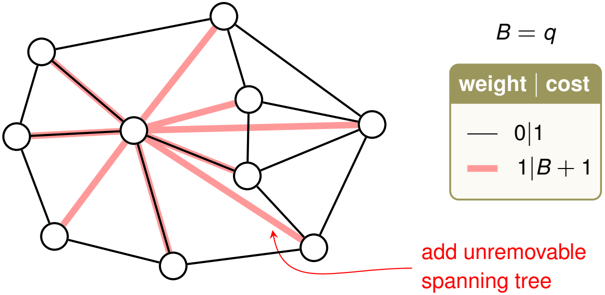

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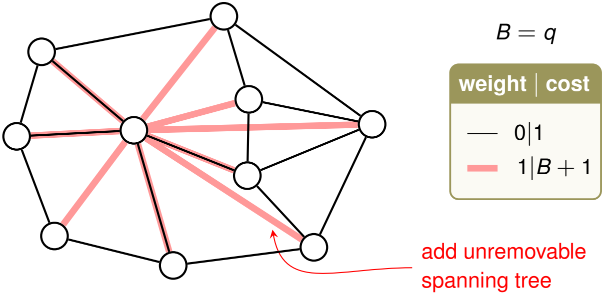

Generalization of MCP: Budgeted Graph Disconnection (BGD)

Break graph into max # of conn. comp. by removing edges of cost $\leq B$.

Best known approximation ratios

MST interdiction	$O(\log m)$	[Frederickson and Solis-Oba, 1996]
k most vital edges	$O(\log k)$	[Frederickson and Solis-Oba, 1996]

- m : # of edges

Previous results

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- m : # of edges

Is there an $O(1)$-approximation for k most vital edges, or even MST interdiction?

Best known running times for most vital edge problem

Deterministic algo	$O(m \cdot \alpha(m, n))$	[Iwano and Katoh, 1993, Chazelle, 2000, Tarjan, 1979]
Randomized algo	$O(m)$	[Iwano and Katoh, 1993, Karger et al., 1995, Dixon et al., 1992]

- m : # of edges
- n : # of vertices
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Further related work on k most vital edges

- Exact exp. time algos (serial and parallel) [Liang and Shen, 1997, Liang, 2001, Bazgan et al., 2011]
- Parameterized complexity [Guo and Shrestha, 2014]

Our results

First $O(1)$-approximation for MST interdiction.

Some consequences

- $O(1)$-approx for metric TSP and TSP path interdiction.

"Remove edges of cost $\leq B$ to make it expensive to visit all vertices."

Follows from $\text{val}(\text{MST}) \leq \text{val}(\text{TSP (path)}) \leq 2\,\text{val}(\text{MST})$.

Some technical hurdles we overcome

- Lack of good relaxation.
- MST interdiction is equivalent to "multi-level" BGD.
  $\rightarrow$ Hard to control interactions between levels.

Some basic observations and notation

Assumption on weights (losing a factor of at most 2)

We assume weights are powers of 2 by rounding weights down.

$\Rightarrow E = E_{-1} \,\dot\cup\, E_0 \,\dot\cup\, E_1 \,\dot\cup\, \dots \,\dot\cup\, E_p.$

$$E_{-1} = \{e \in E \mid w(e) = 0\}$$
$$E_i = \{e \in E \mid w(e) = 2^i\} \text{ for } i \in [p]$$

Assumptions related to E_p

Can assume optimal interdiction set $R \subseteq E$ satisfies $R \cap E_p = \emptyset$.

(Since budget is not enough to disconnect G.)

Can assume $\exists$ interdiction set $R \subseteq E$ s.t. $\mathrm{MST}(V, E \setminus R) \cap E_p \neq \emptyset$.

(Otherwise, delete all edges E_p without changing problem.)

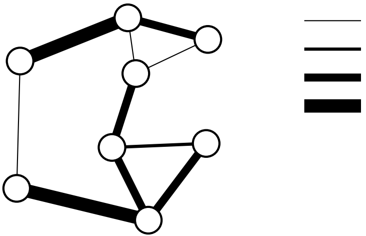
$\Rightarrow$ This implies that opt. value of MST interdiction $\geq 2^p$.

Interpretation as multi-level BGD

$$\text{val(MST)} = \sigma(E_{-1}) - 1 + \sum_{i=0}^{p-1} 2^i \cdot (\sigma(E_{\leq i}) - 1)$$

where for $A \subseteq E$:

$\sigma(A) :=$ # of conn. comp. in (V, A).



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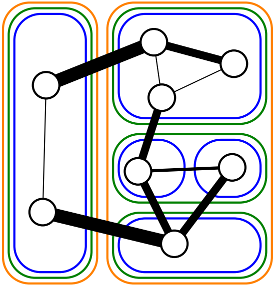

- weight = 0
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- comp. of E_{-1}
- comp. of $E_{\leq 0}$
- comp. of $E_{\leq 1}$

$$\begin{aligned}\text{val}(MST) = & \quad (5 - 1) \\ & +1 \cdot (4 - 1) \\ & +2 \cdot (2 - 1) \\ = & \; 9\end{aligned}$$

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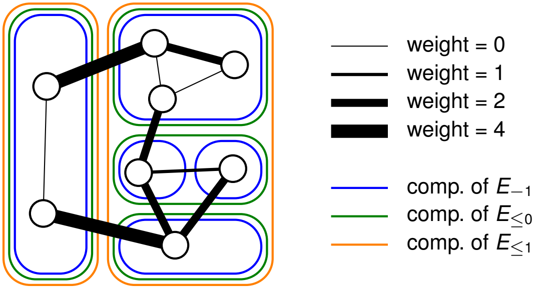

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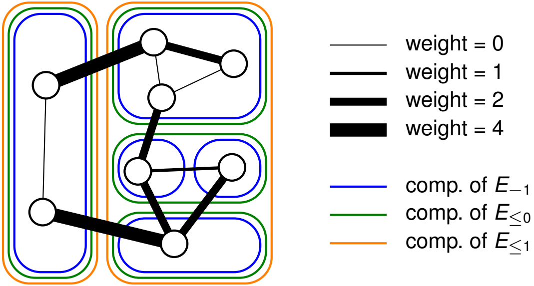

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Goal: $\max\{\mathrm{val}(U) \mid U \subseteq E, c(U) \leq B\}$

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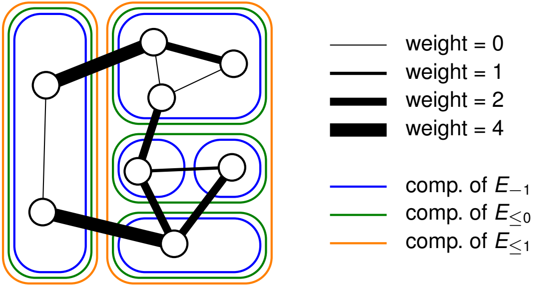

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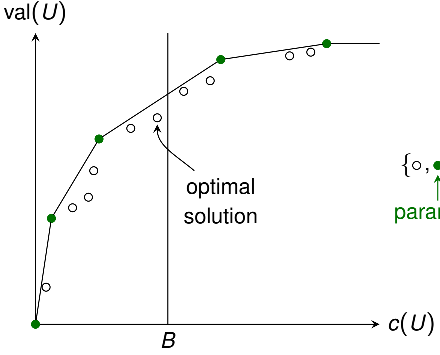
$\max\{\sigma(E_{\leq i} \setminus U) \mid U \subseteq E, c(U) \leq B\}$ is BGD $\forall i$.

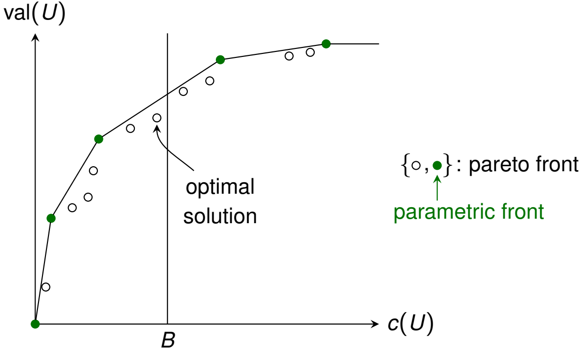
Relaxation through submodular minimization



$\{\circ, \bullet\}$: pareto front

parametric front

Relaxation through submodular minimization



$$\textcolor{red}{\mathrm{val}}(U) = \textcolor{orange}{\sigma}(E_{-1} \setminus U) - 1 + \sum_{i=0}^{p-1} 2^i \cdot (\textcolor{orange}{\sigma}(E_{\leq i} \setminus U) - 1)$$

supermodular (val) — supermodular (σ)

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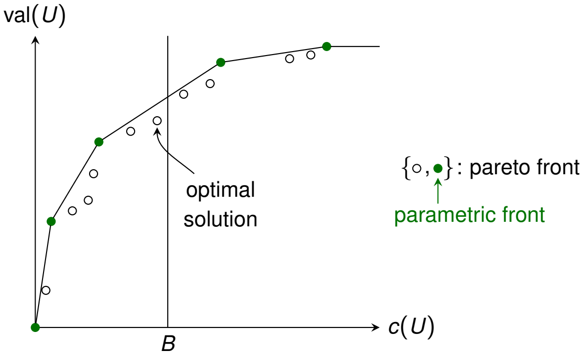

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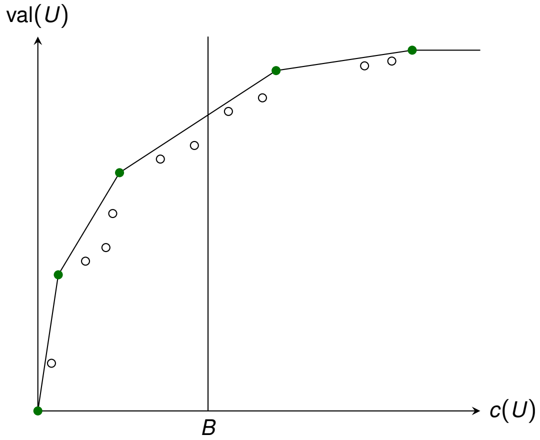
supermodular (val) — supermodular (σ)

Can find parametric frontier efficiently by submodular function minimization (SFM):

Solve $\min\{\lambda \cdot c(U) - \text{val}(U) \mid U \subseteq E_{\leq p-1}\}$ for varying $\lambda \geq 0$.

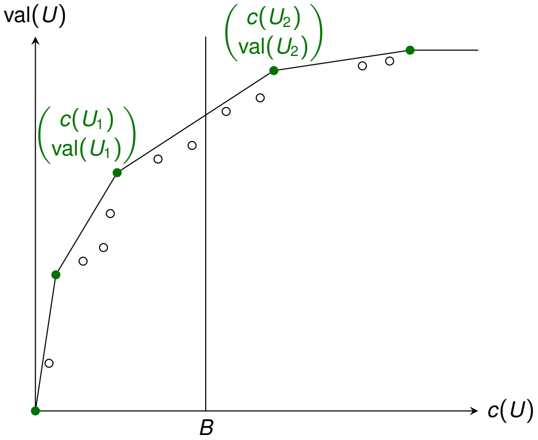


Theorem

For any $U \subseteq E$, $c(U) \geq B$, we can find $R \subseteq U$, $c(R) \leq B$ s.t.

$$\text{val}(R) \geq \frac{1}{6} \cdot B \cdot \frac{\text{val}(U)}{c(U)}.$$

High-level plan



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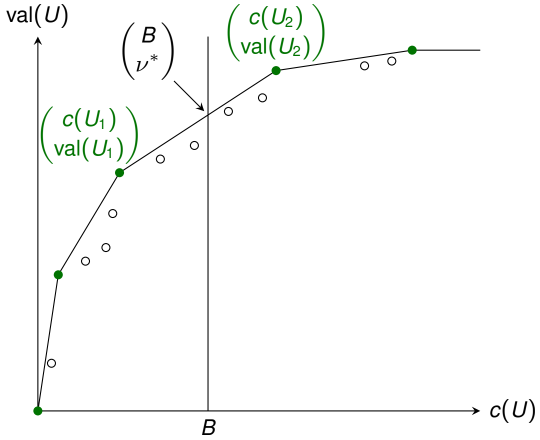

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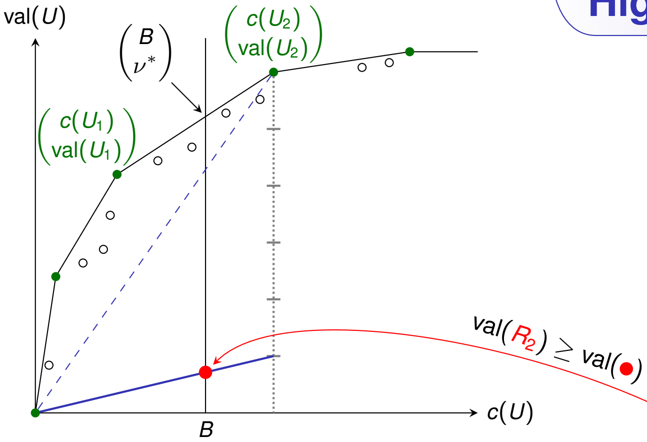

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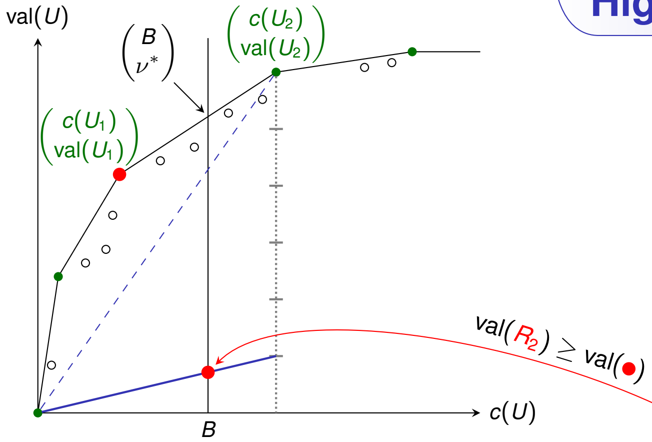

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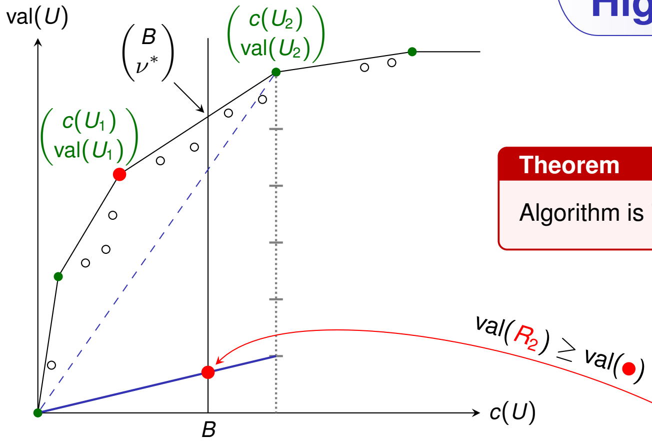

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Algorithm is 7-approximation.

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Goal: exploit/preserve synergies of *U* across levels

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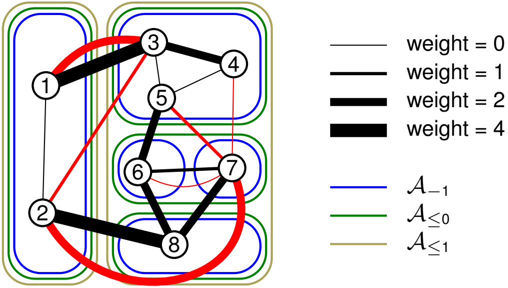

Looking at levels

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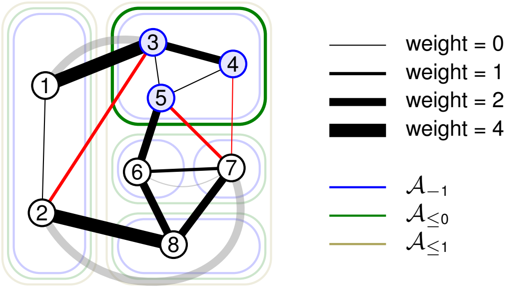
$U_{\leq 0} := U \cap E_{\leq i}$

↓

$\delta(A) \cap U_{\leq 0}$

removing these edges creates comp. A in $\mathcal{A}_0$



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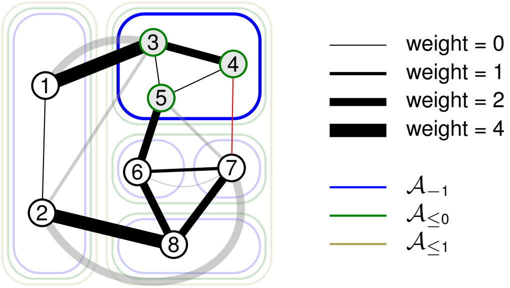
$\delta(A) \cap U_{\leq -1}$

removing these edges creates comp. A in $\mathcal{A}_{-1}$



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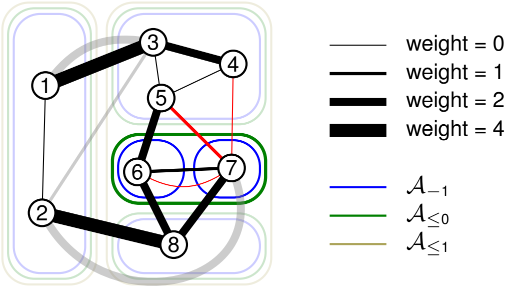
$U_{\leq 0}(A)$

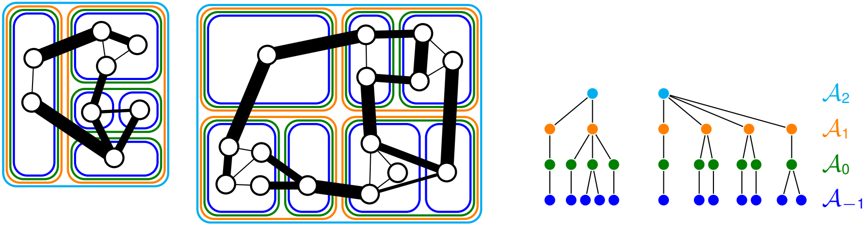
removing these edges creates comp. A in $\mathcal{A}_{-1}$ and all descendant comp.



$$U_{\leq i}(A) := \{e \in U_{\leq i} \mid e \text{ has at least one endpoint in } A\}$$

Picking disjoint comp. in hierarchy

Will chose $(A_1, i_1), \ldots, (A_\beta, i_\beta)$ with $\{A_j \in \mathcal{A}_j\}_j$ disjoint and set $R = \cup_{j=1}^{\beta} U_{\leq i_j}(A_j)$.



- Example of $(V, E \setminus U)$ with edge weights $0, 1, 2, 4, 8$.
- Edges of weight 8 are not shown since wlog they span all vertices.

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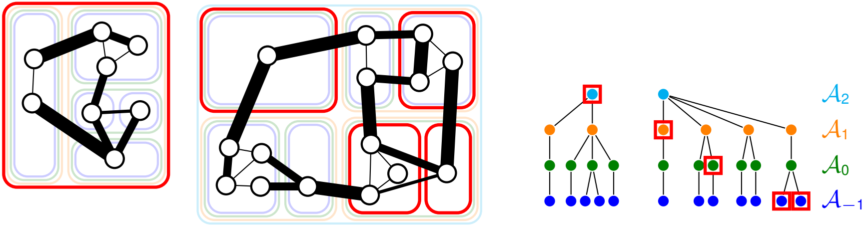

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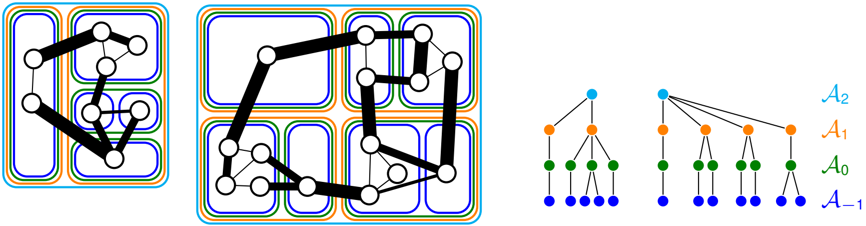
Local measure of cost and impact

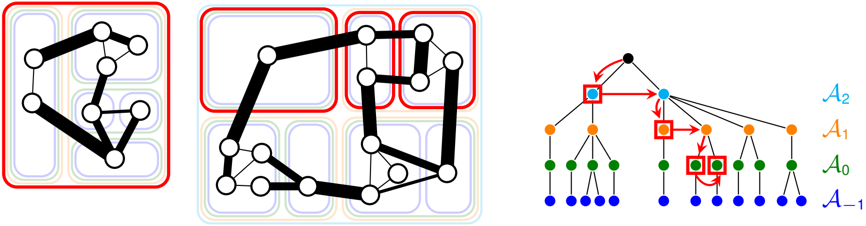
Auxiliary functions to measure things locally for $A \in \mathcal{A}_i$

$$\left.\begin{array}{r}\text{aux. cost: } \kappa_i(A) = \Theta\big(c(U_{\leq i}(A))\big) \\ \text{aux. impact: } g_i(A) = \Theta\left(\sum_{\ell=0}^{i} 2^\ell \cdot |\{D \in \mathcal{A}_\ell \mid D \subseteq A\}|\right)\end{array}\right\} \text{aux. eff.: } \rho_i(A) = \frac{g_i(A)}{\kappa_i(A)}.$$



Sketch of algorithm (traversing tree top down)

- Consider each child (A, i) of current node wrt decreasing auxiliary efficiency.
  - If $c(R \cup U_{\leq i}(A)) \leq B \longrightarrow R = R \cup U_{\leq i}(A)$.
  - Else: move down to first child of (A, i) and repeat.
- Algo stops when either: (i) leaf reached and not selected, or
  (ii) all children of some node are selected.



Ordering of siblings: Siblings further left have higher auxiliary efficencies.

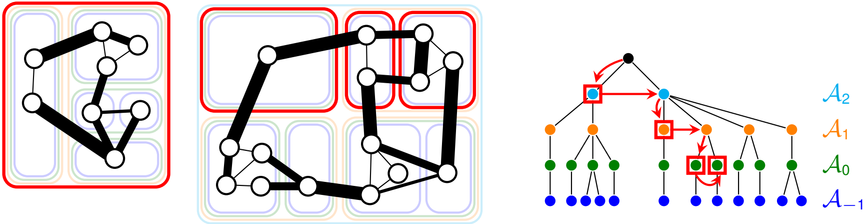
Some aspects of analysis

Average aux. efficiency of children is close to aux. eff. of parent.
→ Error term is gemetrically decreasing and of size $O(2^p)$.

To deal with error term we return best of following two:

(i) constructed set R;
(ii) trivial sol. of value 2^p (making sure one E_p edge is used).

→ Returned set has value: $\geq \frac{1}{6} \cdot B \cdot \frac{\mathrm{val}(U)}{c(U)}$



Summary

- 14-approximation for MST interdiction.
- 28-approximation for metric TSP interdiction.

Many interesting computational questions around interdiction problems are still open.

Some approximation/hardness results

Interdiction of ...	Best known approx. ratio	Hardness result
Shortest paths	-	APX-hardness [Khachiyan et. al, Th Comp Sys '08]
Maximum flow	- (2-pseudoapprox.) [Burch et al., thematic book on int. '03]	strongly NP-hard [Wood, Math & Comp Modeling, '93], [Phillips, STOC'93] "densest k-subgraph hard" [Chestnut & Z., arXiv 15]
Maximum flow on planar graphs	FPTAS [Phillips, STOC'93]	weakly NP-hard
MSTs	$O(\log m)$ [Frederickson & Solis-Oba, SODA'96] $O(1)$ [Z. FOCS'15]	strongly NP-hard [F. & S.-O., SODA'96]
Matchings and some packing problems	$O(1)$ [Dinitz & Gupta, IPCO'13]	strongly NP hard [Z. et al., Disc Math '10]
Connectivity	PTAS [Z., OR Letters '15]	weakly NP-hard
...		

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