k-Trails: Recognition, Complexity, and Approximations

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Introduction

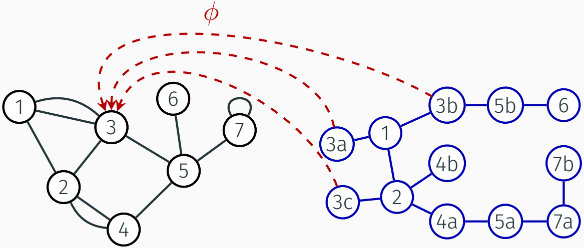
What is a k-trail?

$\exists$ onto fct.: $\phi : W \to V$ s.t. F gets mapped to E.

max degree $\leq k$

Definition

A k-trail $G = (V, E)$ is the homomorphic image of a k-tree $H = (W, F)$.



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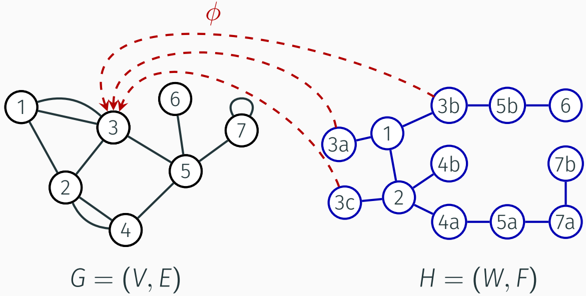

$\Rightarrow$ G is 4-trail

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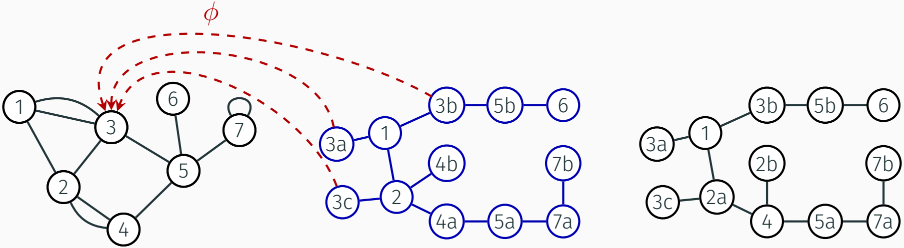

$G = (V, E)$ $H = (W, F)$ $H' = (W', F')$

⇒ G is 4-trail ⇒ G is 3-trail

Motivation and prior results

How it started

- k-trails were introduced by Molnár, Durand, and Merabet [2014a,2014b] (under the name degree-constrained spanning hierarchies).
- Motivation: Use tree being low-degree preimage of a graph to create efficient network routing protocols.

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Some prior results and observations (Molnár, Newman, and Sebő [2015])

- G is a 2-trail $\Leftrightarrow$ G has an Eulerian path.
- Every 2-edge-connected graph is a 3-trail.

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Some interesting questions

- Efficient recognition of k-trail?
- Containment of k-trail: $\exists$? connected subgraph of G that is a k-trail?

Efficient recognition

For any k, one can recognize efficiently whether G is a k-trail.

Containment

- For any $k \geq 3$, deciding whether G contains a k-trail is NP-complete. (extension of proof by Molnár, Durand, and Merabet)
- If G contains k-trail, then it is a $(k+1)$-trail.
- For edge-weighted graphs, if G contains k-trail of weight γ, one can efficiently find a $(2k-1)$-trail of weight $\leq \gamma$.

Our contribution

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Main ingredient: connection to matroids

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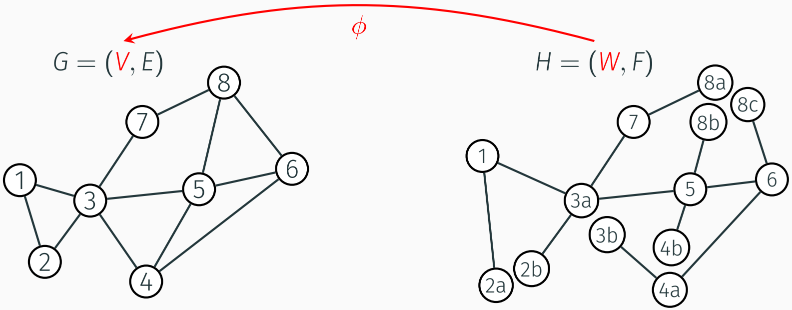
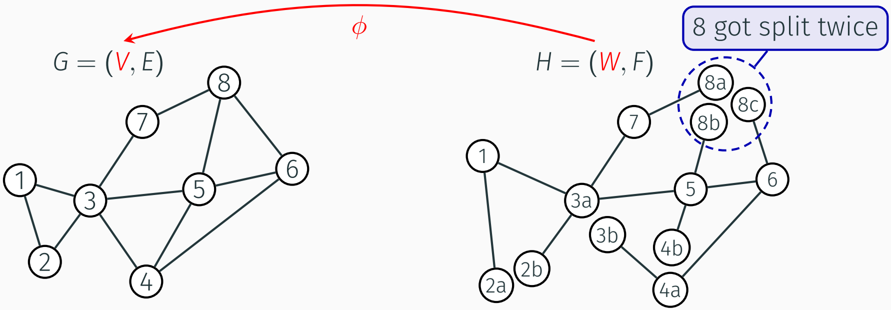
Recognizing k-trails

Balancing degrees





Split vector that corresponds to ϕ: $\alpha = (0, 1, 1, 1, 0, 0, 0, 2)$.

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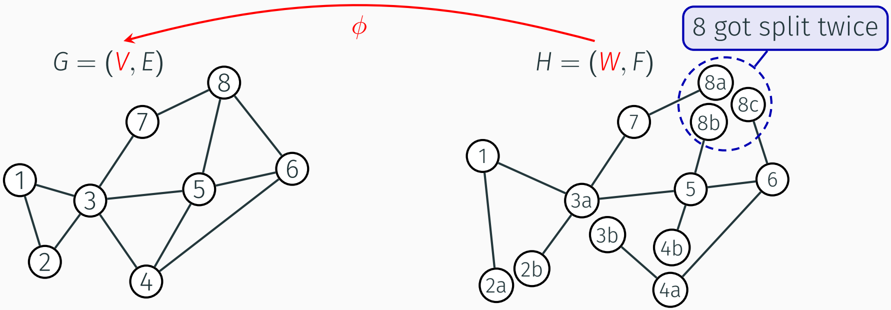

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Balancing Lemma

Given hom. preimage $H = (W, F)$ with split vector α, we can efficiently find another one $H' = (W', F')$ with split vector α and balanced degrees:

$$\deg_{H'}(w') \in \left\{ \left\lfloor \frac{\deg_G(v)}{\alpha_v + 1} \right\rfloor, \left\lceil \frac{\deg_G(v)}{\alpha_v + 1} \right\rceil \right\} \quad \forall \text{ preimage } w' \in W' \text{ of } v \in V.$$

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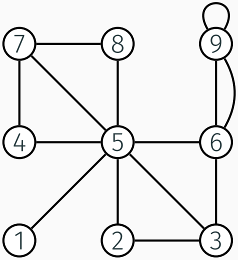

$G = (V, E)$ is k-trail

$$\Longleftrightarrow$$

$\exists$ feasible split vector $\alpha \in \mathbb{Z}^V_{\geq 0}$ with $\alpha_v \geq \frac{\deg_G(v)}{k} - 1 \; \forall v \in V$.

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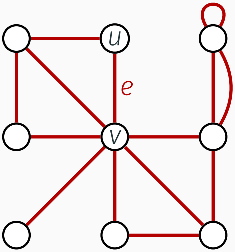
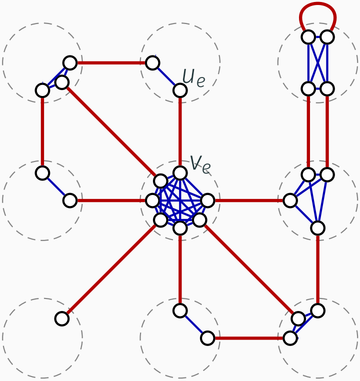
Hom. preimages as spanning trees in an auxiliary graph (1/2)

To check feasible split vectors for $G = (V, E)$, we relate hom. preimages to spanning trees in an auxiliary graph $G' = (V', E')$.

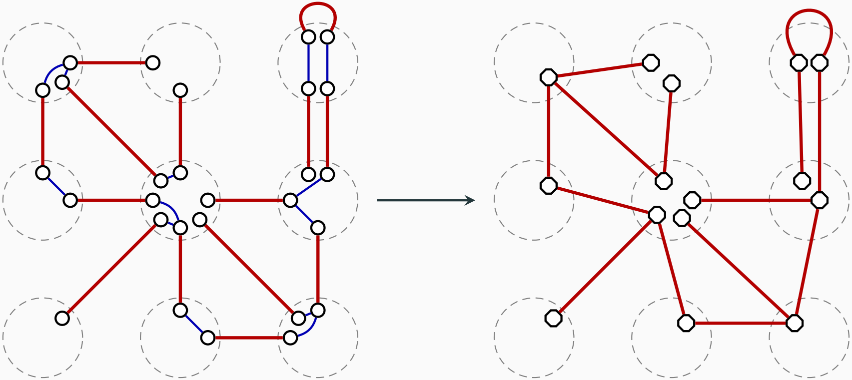


$G = (V, E)$



$G' = (V', E')$

Hom. preimages as spanning trees in an auxiliary graph (2/2)



Spanning trees T in G' containing all ——

contract all —— in T →

Hom. preimages of G.

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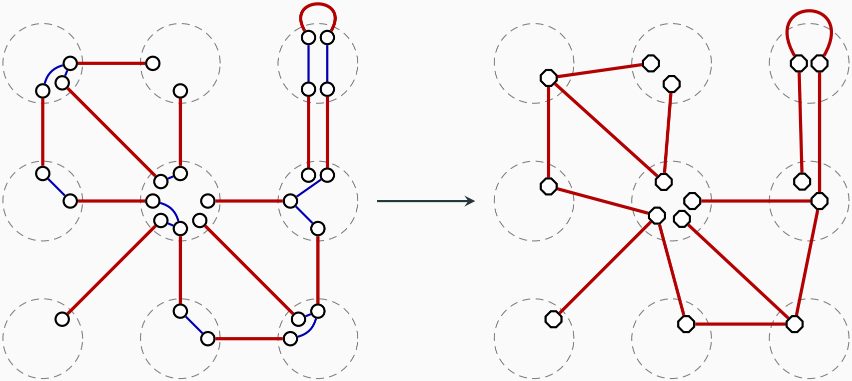

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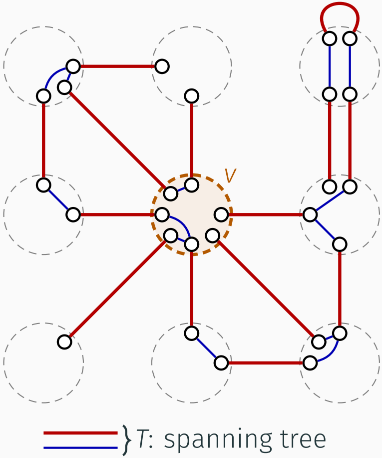
Hom. preimages of G.

This function is onto

k-TRAIL RECOGNITION: PUTTING THINGS TOGETHER



Following quantities are the same

(i) # times v gets split.

(ii) {# of connected comp. of T in ◌} −1.

(iii) $\deg_G(v) - 1 -$ {# of ($T \cap$ —) in ◌}.

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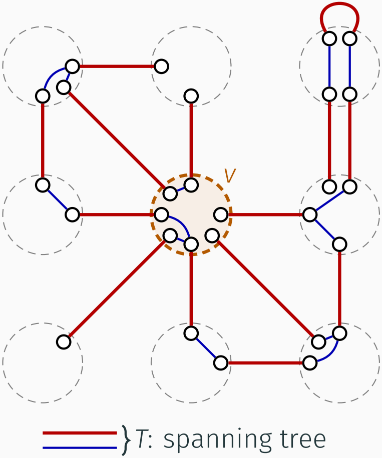

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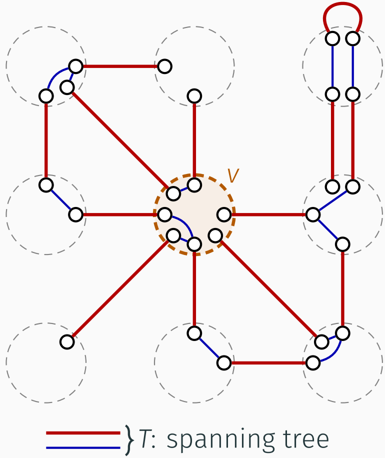

This is a matroid intersection problem!

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- New structures and results:
  - Feasible split vect. are integer points of a polymatroid base polytope.
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We showed $2k-1$ for arbitrary weights.
Maybe, much better guarantees are possible.

References

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