

# A Simple Proof for a Characterization of Sign-Central Matrices using Linear Duality

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# Outline

- 1 Problem definition and motivation
- 2 Complexity of deciding sign-centrality
- 3 Proof of the characterization using linear duality
- 4 A generalization to PSD
- 5 Conclusions

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# Motivation

What can we say about the solutions of a class of linear systems if only the signs of the input matrices are known?

We consider a special case:

Under which conditions do we have that any matrix whose elements satisfy some fixed sign structure, has a non-negative non-zero element in its null-space?

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Under which conditions do we have that any matrix whose elements satisfy some fixed sign structure, has a non-negative non-zero element in its null-space?

## Qualitative classes

### Definition (Sign of a matrix)

With any matrix  $A \in \mathbb{R}^{m \times n}$  we associate a sign matrix  $\text{sign}(A) \in \{0, +, -\}^{m \times n}$ :

$$\begin{pmatrix} 1 & -3 & 4 & 0 & -5 \\ 2 & 0 & -6 & 1 & -3 \\ 0 & -2 & 0 & 3 & 8 \end{pmatrix} \xrightarrow{\text{sign}} \begin{pmatrix} + & - & + & 0 & - \\ + & 0 & - & + & - \\ 0 & - & 0 & + & + \end{pmatrix} .$$

### Definition (Qualitative class)

The qualitative class  $Q(A)$  of a matrix  $A \in \mathbb{R}^{m \times n}$  are the set of all matrices with the same sign as  $A$ , i.e.,

$$Q(A) := \{B \in \mathbb{R}^{m \times n} \mid \text{sign}(B) = \text{sign}(A)\} .$$

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# Central and sign-central matrices

## Definition (Central matrix)

$A \in \mathbb{R}^{m \times n}$  is central if there exists a nonnegative, non-zero element in its null-space, i.e.,  $\{x \in \mathbb{R}^n \mid Ax = 0, x \geq 0, x \neq 0\} \neq \emptyset$ .

## Definition (Sign-central matrix)

$A \in \mathbb{R}^{m \times n}$  is sign-central if every matrix in  $Q(A)$  is central.

How to characterize sign-central matrices?



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How to characterize sign-central matrices?

## Closed matrices

### Definition (Closed matrix)

$A \in \mathbb{R}^{m \times n}$  is closed if for any non-strict sign vector  $v \in \{\oplus, \ominus\}^m$ , there is a column in  $A$  whose sign corresponds to  $v$ , i.e.,  $\exists j \in \{1, \dots, n\}$  with

$$\text{sign}(A)_{ij} = \begin{cases} 0 \text{ or } + & \text{if } v_i = \oplus \\ 0 \text{ or } - & \text{if } v_i = \ominus \end{cases} \quad \forall i \in \{1, \dots, m\} .$$

## Closed matrices: Example

$$A = \begin{pmatrix} 1 & -3 & 4 & 0 & -5 \\ 2 & 0 & -6 & 1 & -3 \\ 0 & -2 & 0 & 3 & 8 \end{pmatrix}, \quad \text{sign}(A) = \begin{pmatrix} + & - & + & 0 & - \\ + & 0 & - & + & - \\ 0 & - & 0 & + & + \end{pmatrix}$$

The matrix  $A$  is closed since we find for all of the vectors

$$\begin{pmatrix} \oplus \\ \oplus \\ \oplus \end{pmatrix}, \begin{pmatrix} \oplus \\ \oplus \\ \ominus \end{pmatrix} \dots \begin{pmatrix} \ominus \\ \ominus \\ \ominus \end{pmatrix}$$

a corresponding column in  $\text{sign}(A)$ .

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# A characterization of sign-central matrices

## Theorem ([Davidov and Davidova, 1992])

Let  $A \in \mathbb{R}^{m \times n}$ .

$A$  is sign-central  $\Leftrightarrow A$  is closed .

- This theorem was first proven in [Davidov and Davidova, 1992] with a long and rather complicated proof.
- In [Ando and Brualdi, 1994] an alternative proof was presented using induction and a separation theorem of convex sets.
- Both proofs are hard to adapt to similar problems.



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- Both proofs are hard to adapt to similar problems.

We present a simple, short proof based on LP duality.

## Why to bother about a new proof?

- The proof shows how **duality theory** can be used **in the context of sign-systems** → well-established theory of convex duality can be used to tackle problems dealing with sign-systems.
- The approach used in the proof can be adapted for **proving other characterization theorems** (we will show an extension to PSD).
- The presented proof is very short and allows easily to get a **good understanding of the characterization**.

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## Equivalence with tautologic expressions

With every matrix  $A \in \mathbb{R}^{m \times n}$  we associate a Boolean expression  $E(A)$  as shown below.

$$\text{sign}(A) = \begin{pmatrix} + & - & + & 0 & - \\ + & 0 & - & + & - \\ 0 & - & 0 & + & + \end{pmatrix} \quad \begin{array}{l} \rightsquigarrow x_1 \\ \rightsquigarrow x_2 \\ \rightsquigarrow x_3 \end{array} \quad \boxed{\begin{array}{l} \text{true: } \oplus \\ \text{false: } \ominus \end{array}}$$

$$E(A) = (x_1 \wedge x_2) \vee (\bar{x}_1 \wedge \bar{x}_3) \vee (x_1 \wedge \bar{x}_2) \vee (x_2 \wedge x_3) \vee (\bar{x}_1 \wedge \bar{x}_2 \wedge x_3)$$

### Property

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## Link with SAT problem

### Property

$A$  is not sign-central  $\Leftrightarrow \neg E(A)$  allows a satisfying truth assignment .

Since  $\neg E(A)$  is a *SAT*-problem in its general form we have:

### Theorem

$\rightarrow$  *Deciding whether a matrix is sign-central is co-NP-complete.*

### Remark

However, many interesting special cases are easy to solve, e.g., when the given matrix has a bounded number of zero entries per column.

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## Preliminaries

Let  $A \in \mathbb{R}^{m \times n}$ . We define the following dual pair of LPs:

$$LP(A) \quad \begin{array}{ll} \max & e^T x \\ \text{s.t.} & Ax = 0 \\ & x \geq 0 \end{array}$$

$$DP(A) \quad \begin{array}{ll} \min & 0 \\ \text{s.t.} & A^T y \geq e \\ & y \text{ free} \end{array},$$

where  $e = (1, \dots, 1) \in \mathbb{R}^n$ .

### Property 1

$A$  is central  $\Leftrightarrow OPT(LP(A)) = \infty \Leftrightarrow DP(A)$  is infeasible .

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## A sign-central $\Rightarrow$ A closed (1)

Suppose  $A$  not closed  $\Rightarrow \exists v \in \{\oplus, \ominus\}^m$  s.t. no column of  $A$  corresponds to  $v$ .

We will define a matrix  $B$  and a vector  $y \in \mathbb{R}^m$  satisfying  $\text{sign}(B) = \text{sign}(A)$  and  $B^T y \geq e$  ( $\rightarrow$   $DP(B)$  is feasible).

We choose

$$y_i = \begin{cases} 1 & \text{if } v_i = \ominus \\ -1 & \text{if } v_i = \oplus \end{cases} \quad \forall i \in \{1, \dots, m\} .$$

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## A sign-central $\Rightarrow$ A closed (2)

We start with  $B = A$  and modify  $B$  to satisfy  $B^T y \geq e$ . Let  $j \in \{1, \dots, n\}$  and consider  $(B^T y)_j$ . Since  $B$  has no column whose sign structure is opposite to  $y$ , we have

$$\exists i \in \{1, \dots, m\} \text{ with } B_{ij} \cdot y_i > 0 .$$

By choosing  $|B_{ij}|$  large enough we ensure  $(B^T y)_j \geq 1$ . By repeating the process for every  $j \in \{1, \dots, n\}$  we finally get

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## $A$ sign-central $\Leftrightarrow A$ closed

By contradiction we assume that  $A$  is not sign-central but closed.

$$\begin{array}{l} \xrightarrow{A \text{ not sign-central}} \\ \xrightarrow{\text{by Property 1}} \end{array} \quad \begin{array}{l} \exists B \in \mathbb{R}^{n \times m} \text{ not central with } \text{sign}(B) = \text{sign}(A). \\ \exists y \in \mathbb{R}^m \text{ with } B^T y \geq e. \end{array}$$

Since  $B$  is closed,  $B$  contains a column whose sign structure is opposite to  $y$ , i.e.,

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This is in contradiction with  $B^T y \geq e$ . □



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## PSD sign-centrality

### Definition (PSD central matrix family)

We call a family of matrices  $\{A^k\}_{k \in \{1, \dots, r\}} \subset \mathbb{R}^{n \times n}$  PSD central if

$$\{X \succeq 0 \mid A^k \bullet X = 0 \quad \forall k \in \{1, \dots, r\}, X \neq 0\} \neq \emptyset .$$

### Definition (PSD sign-central matrix family)

A family  $\{A^k\}_{k \in \{1, \dots, r\}} \subset \mathbb{R}^{n \times n}$  is called PSD sign-central if every family  $\{B^k\}_{k \in \{1, \dots, r\}} \subset \mathbb{R}^{n \times n}$  with  $\text{sign}(B_k) = \text{sign}(A_k) \forall k$  is central.

### Remark

The above definition for sign-centrality of PSD is a **generalization** of classical sign-centrality:

→ Choose all  $A^k$  to be diagonal matrices.

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## A characterization for PSD sign-centrality

Again we can define an appropriate pair of primal and dual problems:

$LP(\{A^k\})$

$$\begin{array}{ll} \max & I \bullet X \\ \text{s.t.} & A^k \bullet X = 0 \quad \forall k \\ & X \succeq 0 \end{array}$$

$DP(\{A^k\})$

$$\begin{array}{ll} \min & 0 \\ \text{s.t.} & \sum_{k=1}^r \lambda_k A^k \succeq I \\ & \lambda_k \text{ free } \forall k \end{array}$$

where  $I \in \mathbb{R}^{n \times n}$  is the identity matrix.

### Theorem

$$\{A^k\} \text{ is PSD sign-central} \Leftrightarrow \begin{pmatrix} \text{diag}(A^1)^T \\ \vdots \\ \text{diag}(A^r)^T \end{pmatrix} \text{ is closed .}$$

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## Conclusions

- New approach for proving characterizations of sign-systems:
  - Based on **duality theory**.
  - **Simple, short proof** of a classical characterization.
  - Underlines that the characterization is essentially a **theorem of alternatives**.
- The approach can be adapted to **profit from convex duality theory** for proving **more general characterizations** – Example: PSD sign-centrality.
- We hope that the shown link to duality theory allows to **find and prove further results** for sign-systems.

## References I

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