Eidgenössische Technische Hochschule Zürich
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# A bound on the collection of edges in MST's of fixed-size subgraphs

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# Outline

1. **Introduction**
   - The conjecture
   - Interpretation using $k$-constructible graphs
   - Related work
2. **Proof of the conjecture**
   - Inductive approach
   - Splitting into $G^1$ and $G^2$

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# Conjecture of Goemans and Vondrák

## Definition

*Let $G = (V, E, w)$ be a complete n-vertex graph with distinct positive edge weights $w$. For $k \in \{1, 2, \ldots, n-1\}$ We define*

$$M_k(G) = \bigcup_{X \subseteq V, |X| = k-1} MST(G \setminus X)\,.$$

## Theorem (Conjecture of Goemans and Vondrák)

$$|M_k(G)| \leq nk - \binom{k+1}{2}$$

# Outline

# Menger's theorem

In further discussions we regulary use a special form of Menger's theorem.

## Definition

Two vertices $s, t$ in an undirected graph $G$ are called $k$-connected if there exist $k$ vertex disjoint paths connecting them in $G$.

## Theorem

*Let $s, t$ be two non-adjacent vertices in an undirected graph $G$. $s$ and $t$ are $k$-connected in $G$ if and only if after deleting $k - 1$ vertices (distinct from $s$ and $t$) $s$ and $t$ are still connected.*

# Which edges are in $M_k(G)$?

## Some notations

- $E = \{e_1, e_2, \ldots, e_m\}$ with $w(e_1) < w(e2) < \cdots < w(e_m)$.
- $G_i = (V, \{e_1, e_2, \ldots, e_i\})$.

**Theorem**
*An edge $e_i \in E$ is exactly then in $M_k(G)$ when $G_{i-1}$ contains at most $k-1$ vertex disjoint paths between the two endpoints of $e_i$.*

$\rightarrow$ direct consequence of Menger's theorem

# $k$-constructible graphs

## Definition

A graph $G = (V, E)$ is called $k$-constructible if there exist an ordering $O = (e_1, e_2, \dots, e_{|E|})$ of the edges in $E$ such that for all $i \in \{1, 2, \dots, m\}$ the graph $(V, \{e_1, e_2, \dots, e_i\})$ contains at most $k$ vertex disjoint paths between the two endpoints of $e_i$. The order $O$ is called a $k$-construction order for $G$. The triple $(V, E, O)$ is called a $k$-construction.

- $\mathcal{G}_{n,k} :=$ set of all $k$-constructible graphs with $n$ vertices

# $k$-constructible graphs (continued)

**Theorem**

*For every complete graph $G = (V, E, w)$ with $n$ vertices and distinct positive edge weights $w$ we have*

$$M_k(G) \in \mathcal{G}_{n,k} \quad \forall k \{1, 2, \ldots, n-1\} .$$

We will prove the following theorem implying the conjecture of Goemans and Vondrák.

**Theorem**

*For $G = (V, E) \in \mathcal{G}_{n,k}$ we have*

$$|E| \leq nk - \binom{k+1}{2} .$$

# $k$-constructible graphs (continued)

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# $k$-minimal graphs

## Definition

A graph $G = (V, E)$ is a $k$-minimal graph if it is $k$-connected and removing any edge of $G$ destroys $k$-connectivity.

## Property

*Every $k$-minimal graph is $k$-constructible (any order of its edges is a $k$-construction order) but not the converse.*

## Theorem (Mader 1971)

*Every $k$-minimal graph $G = (V, E)$ with $n$ vertices satisfies*

$$|E| \leq nk - \binom{k+1}{2}.$$

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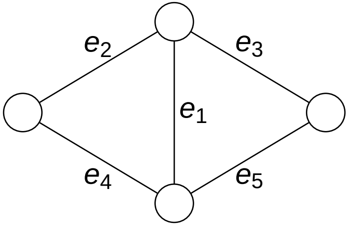
# Example: $k$-constructible graph not being $k$-minimal

The graph below is 2-constructible with 2-construction order $(e_1, e_2, e_3, e_4, e_5)$ but eliminating the edge $e_1$ does not destroy 2-connectivity.



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# Inductive hypothesis and base step

We fix $k \in \mathbb{N}$.

## Inductive hypothesis

The conjecture is true for $k$-constructible graphs with number of vertices $\in \{k+1, k+2, \ldots, n-1\}$.

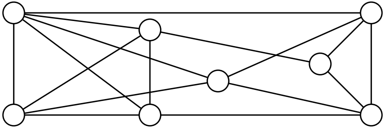
## Base step

For graphs $G = (V, E)$ with $k+1$ vertices, the conjecture states

$$|E| \leq \binom{k+1}{2}$$

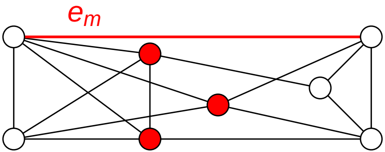
and is thus trivially true.

# Splitting into two $k$-constructible graphs



$G = (V, E)$, a 4-constructible graph

# Splitting into two $k$-constructible graphs



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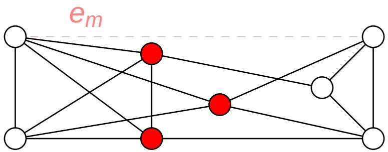

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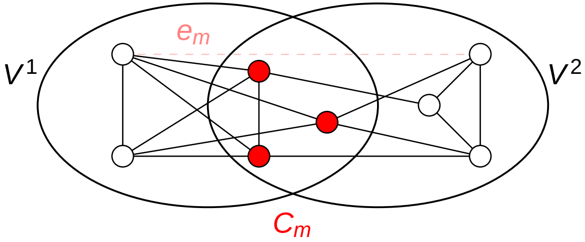

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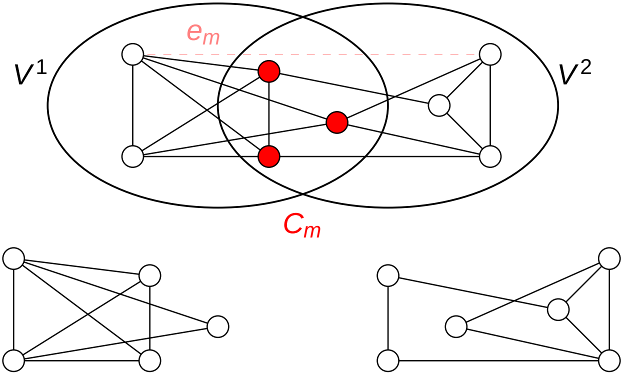
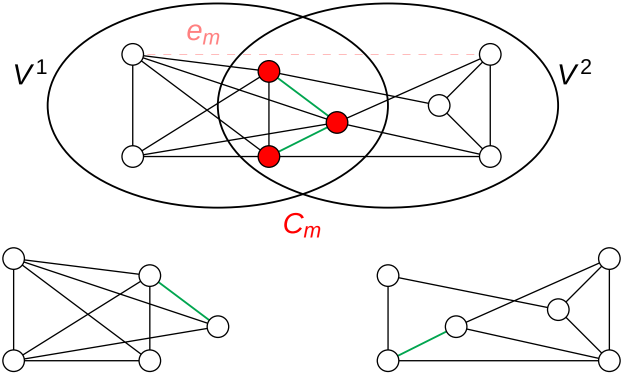


# Splitting into two $k$-constructible graphs

$$B = (C_m \times C_m) \setminus E$$



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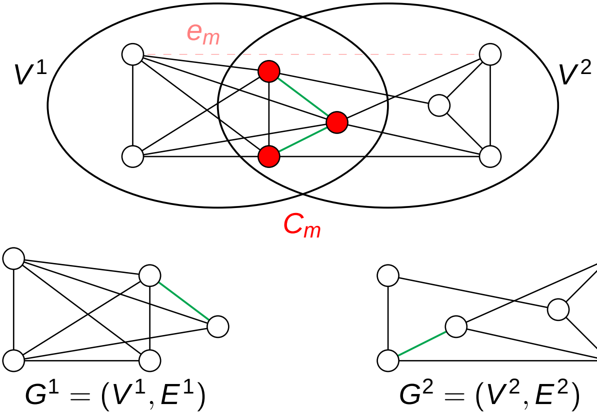

# Applying the inductive hypothesis

$$|E^1| + |E^2| = \underbrace{(|E| - 1)}_{\text{I}} + \underbrace{|G[C_m]|}_{\text{II}} + \underbrace{\left(\binom{k-1}{2} - |G[C_m]|\right)}_{\text{III}}$$

The terms in the above sum represent:

- I : the number of edges in $G$ without $e_m$
- II : the double-counting of the edges $G[C_m]$
- III : the number of added edges in $C_m \times C_m$

# Applying the inductive hypothesis

$$|E^1| + |E^2| = (|E| - 1) + |G[C_m]| + \left( \binom{k-1}{2} - |G[C_m]| \right)$$

$$\Rightarrow |E| = 1 + |E^1| + |E^2| - \binom{k-1}{2}$$

# Applying the inductive hypothesis

$$|E^1| + |E^2| = (|E| - 1) + |G[C_m]| + \left(\binom{k-1}{2} - |G[C_m]|\right)$$
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Applying the inductive hypothesis on $G^1$ and $G^2$, we finally get

$$|E| \leq 1 + \left(|V^1|k - \binom{k+1}{2}\right) + \left(|V^2|k - \binom{k+1}{2}\right) - \binom{k-1}{2}$$
$$= nk - \binom{k+1}{2}.$$

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# Choice of $G$

It suffices to show the correctness of the conjecture for $G = (V, E) \in \mathcal{G}_{n,k}$ having the maximum number of edges among all graphs in $\mathcal{G}_{n,k}$.

## Notations

- $O = (e_1, e_2, \dots, e_m)$: a $k$-construction order for $G$.
- $G_i = (V, \{e_1, e_2, \dots, e_i\})$.
- $r_1^j < r_2^j < \cdots < r_{m^j}^j$: indices of the edges in $E$ having both endpoints in $V^j \quad \forall j \in \{1, 2\}$

In the next steps we will build two $k$-constrution orders $O^1$,$O^2$ corresponding to $G^1$ and $G^2$. The order $O^j$ be built by starting with $(e_{r_1^j}, e_{r_2^j}, \dots, e_{r_{m^j}^j})$ and adding edges of $B$ into this order.

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# Partitioning $B$

We partition $B(= (C_m \times C_m) \setminus E))$ into the following sets $B_1, B_2, \ldots, B_m$.

$$B_i = \left\{ \{v_1, v_2\} \in B \,\middle|\, \begin{array}{l} v1 \text{ and } v2 \text{ are } k\text{-connected in } G_i \text{ and} \\ k-1\text{-connected in } G_{i-1} \end{array} \right\}$$

## Definition

$\widetilde{G}_i := (V, \{e_1, e_2, \ldots, e_i\} \cup \bigcup_{l=1}^{i} B_l)$

## Theorem (A)

*For all $i \in \{1, 2, \ldots, m\}$ we have that the two endpoints of $e_i$ are at most $k-1$-connected in $\widetilde{G}_{i-1}$.*

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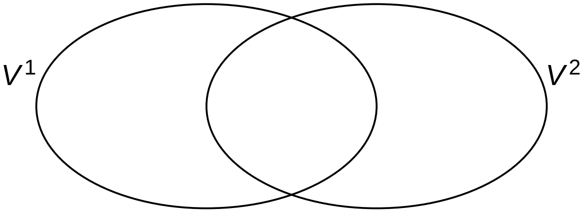
# Proof of theorem (A)

- Let $C_i \subseteq V$ with $|C_i| = k-1$ be a set separating the two endpoints of $e_i$ in $G_{i-1}$.
- $\forall e \in \bigcup_{l=1}^{i} B_l$, $C_i$ cannot separate the two endpoints of $e$ in $G_{i-1}$ because they are at least $k$-connected.
- Thus the edges in $\bigcup_{l=1}^{i} B_l$ only connect vertices in $G_{i-1}[V \setminus C_i]$ which are already in the same connected component.
- $C_i$ is a separating set for the two endpoints of $e_i$ in $\widetilde{G}_i$.

# Structure of $O^j$

$$O = (e_1, e_2, \ldots, e_m)$$



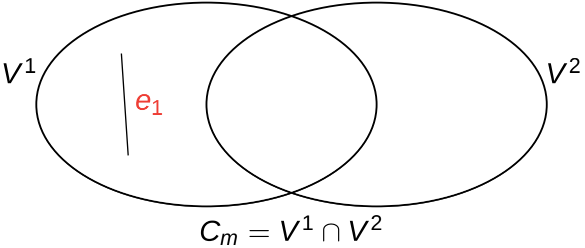

$$C_m = V^1 \cap V^2$$

Constructing $O^1$
()

Constructing $O^2$
()

# Structure of $O^j$

$$O = (e_1, e_2, \ldots, e_m)$$




$\rightarrow$ Partition of $B_1$ into $B_1^1, B_1^2$

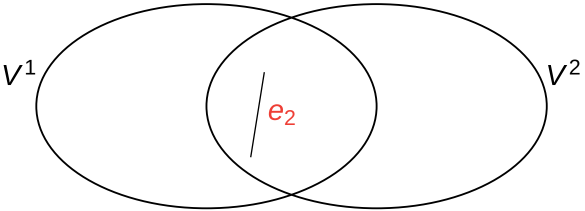
$\rightarrow$ Define order $O_1^j$ for elements in $B_1^j$

Constructing $O^1$
$(e_1, O_1^1)$

Constructing $O^2$
$(O_1^2)$

# Structure of $O^j$

$O = (e_1, e_2, \ldots, e_m)$




$C_m = V^1 \cap V^2$

$\rightarrow$ Partition of $B_2$ into $B_2^1, B_2^2$

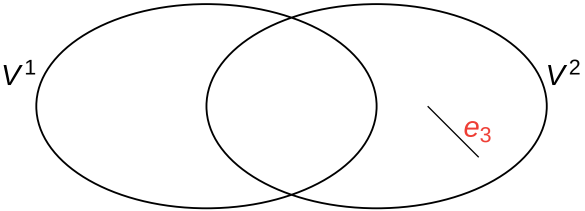
$\rightarrow$ Define order $O_2^j$ for elements in $B_2^j$

Constructing $O^1$
$(e_1, O_1^1, e_2, O_2^1)$

Constructing $O^2$
$(O_1^2, e_2, O_2^2)$

# Structure of $O^j$

$$O = (e_1, e_2, \ldots, e_m)$$




$$C_m = V^1 \cap V^2$$

$\rightarrow$ Partition of $B_3$ into $B_3^1, B_3^2$

$\rightarrow$ Define order $O_3^j$ for elements in $B_3^j$

Constructing $O^1$
$(e_1, O_1^1, e_2, O_2^1, O_3^1)$

Constructing $O^2$
$(O_1^2, e_2, O_2^2, e_3, O_3^2)$

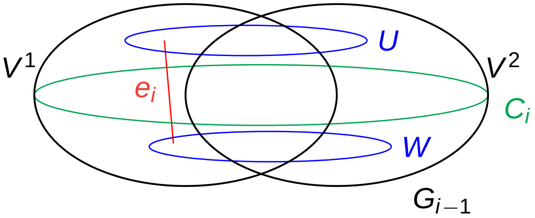
# Structure of $B_i$

We fix $i \in \{1, 2, \ldots, m\}$ and define

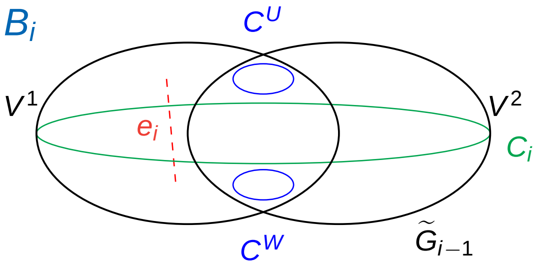
- $C_i \subset V$ with $|C_i| = k - 1$: a set separating the two endpoints of $e_i$ in $G_{i-1}$.
- $U, W \subseteq V \setminus C_i$: the two components in $G_{i-1}[V \setminus C_i]$ containing the two endpoints of $e_i$
- $C^U = C_m \cap U$, $C_m^W = C \cap W$

## Property

$B_i \subseteq C^U \times C^W$

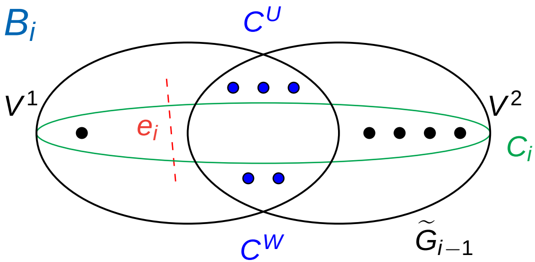


# Splitting $B_i$




- $S^j = C_i \cap V^j$ separates the two endpoints of $e_i$ in $\widetilde{G}_{i-1}[V^j]$
- We can add any $k - 1 - |S^j|$ edges to the current graph $j$ without violating $k$-constructibility
- $\max_{j \in \{1,2\}}\{k - 1 - |S^j|\} \geq \min\{|C^U|, |C^W|\}$

# Splitting $B_i$




- Suppose $|C^U| \geq |C^W|$, and $|V^1| \leq |V^2|$. Choose $v \in C^U$.
- Add all edges of $B_i$ being adjacent to $v$ in any order to $O_i^1$
- $S^1 \cup \{v\}$ separates all remaining edges in $B_i$ in the current graph over $V^1$ and $S^2$ still separates all remaining edges of $B_i$ in the graph over $V^2$.
- Repeat the previous steps over the remaining edges in $B_i$ until all edges are associated to one of the two orders $O_i^1$,$O_i^2$.