A Flow Model Based on Polylinking Systems

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Joint work with Michel Goemans and Satoru Iwata

August 25, 2009

Outline

1. Motivation (wireless information flow)
2. A flow model based on (poly-)linking systems
3. Conclusions

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1. **Motivation (wireless information flow)**
2. A flow model based on (poly-)linking systems
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Wireless information flows

Features of wireless information flows

- Broadcasting (signal emitted by one transmitter is received by many nodes).
- Superposition of signal (interference).

⇒ This leads to complex signal interactions.

Classical model: Multiuser Gaussian Channel

- Unknown how the capacity of the network can be determined except for simplest networks.

The ADT model [Avestimehr, Diggavi, and Tse, 2007a]

- A deterministic model to approximate Multiuser Gaussian Channels.

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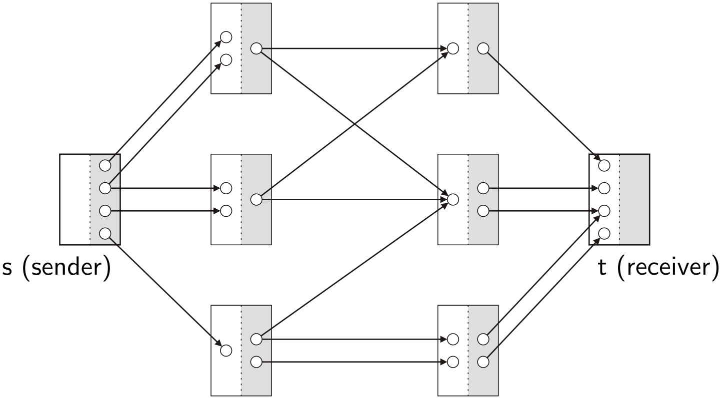

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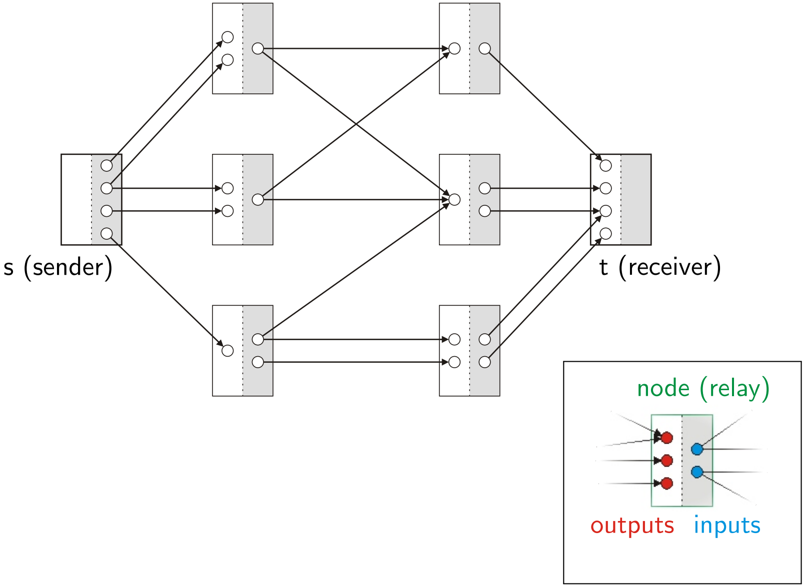

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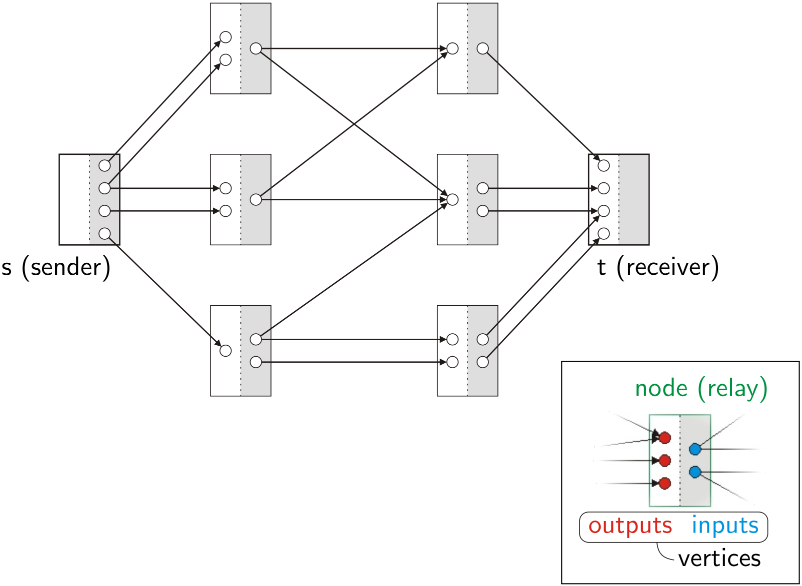

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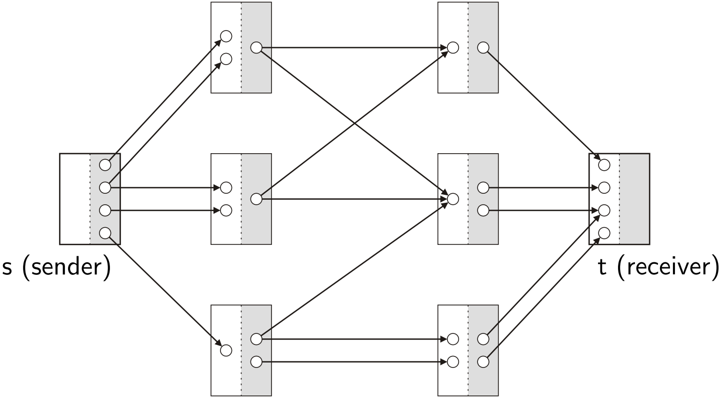

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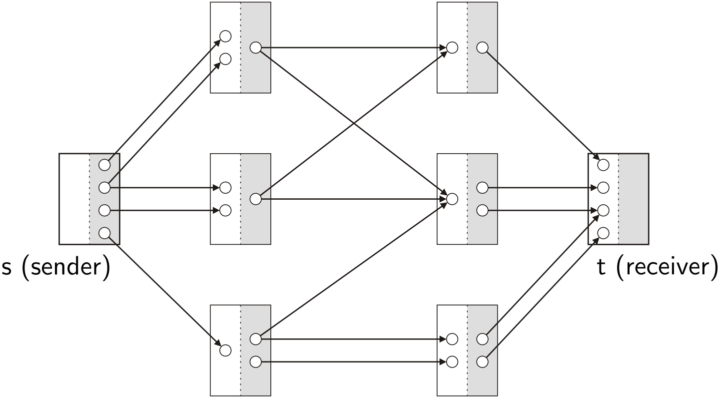

- Task: Send maximum number of signals from s to t.
- A signal is an element of $\mathbb{F}_2$.

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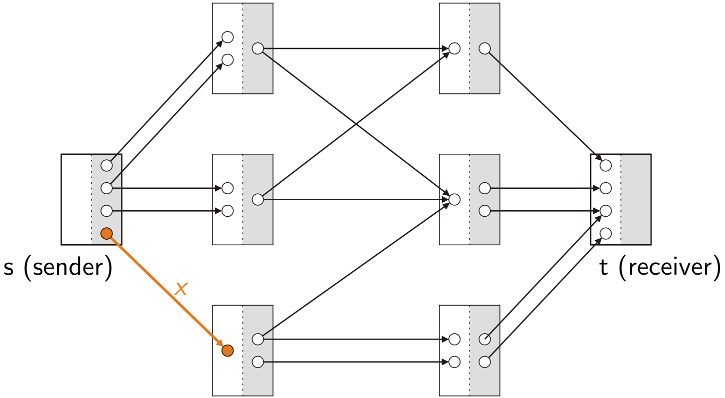

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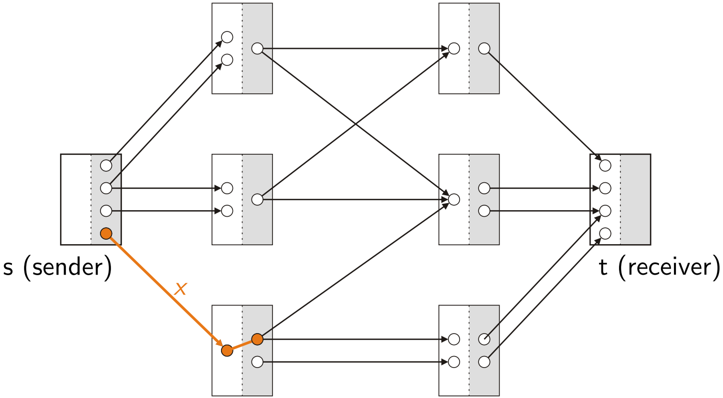

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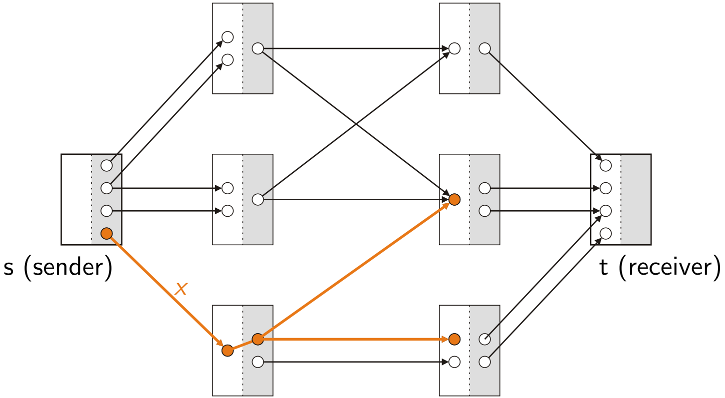

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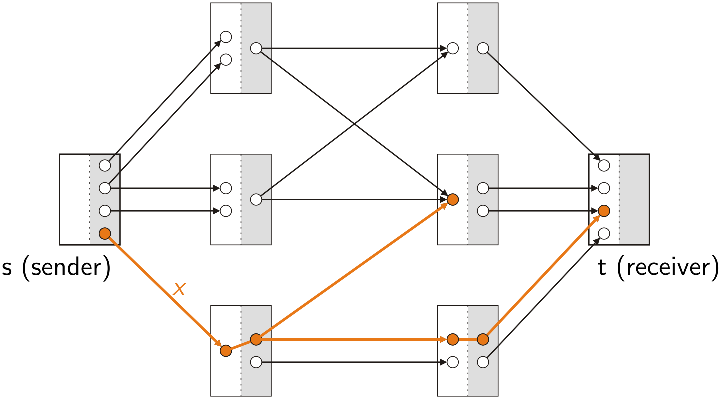

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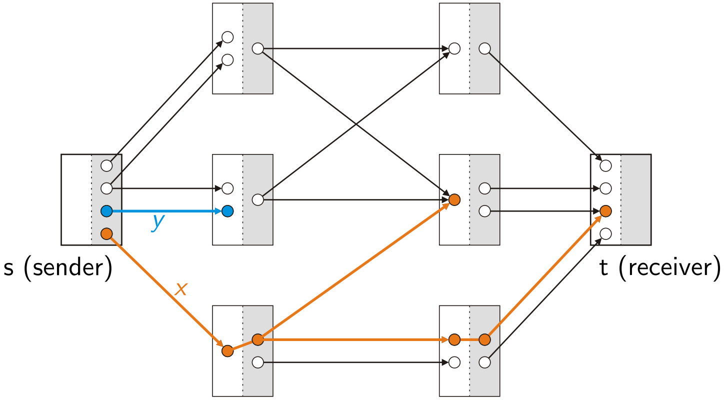

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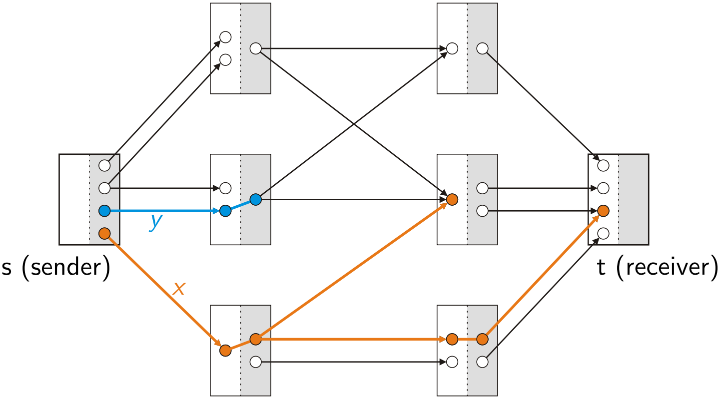

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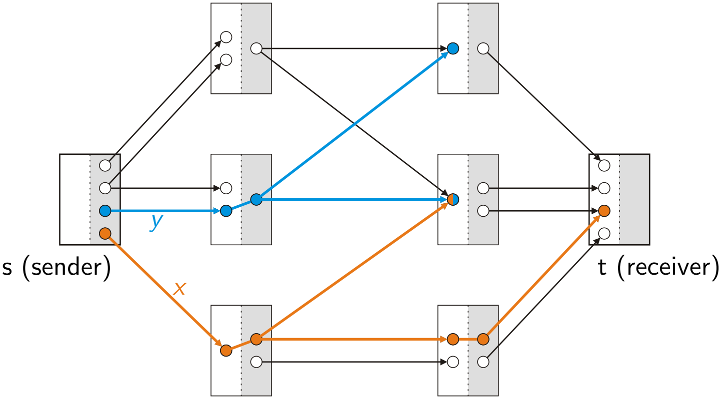

- $\rightarrow$ Interference between the two signals!
- Interference is modelled as XOR.

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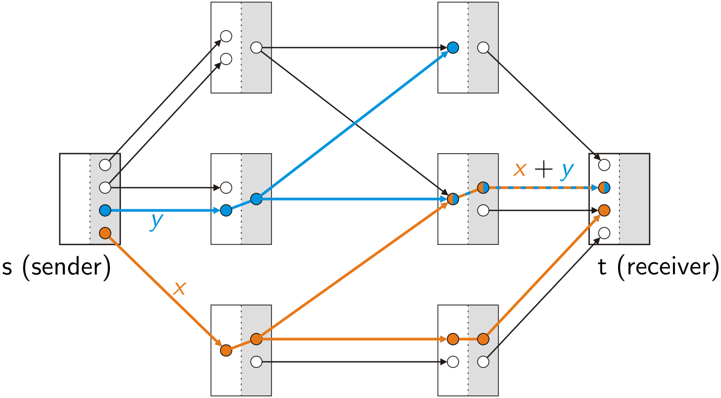

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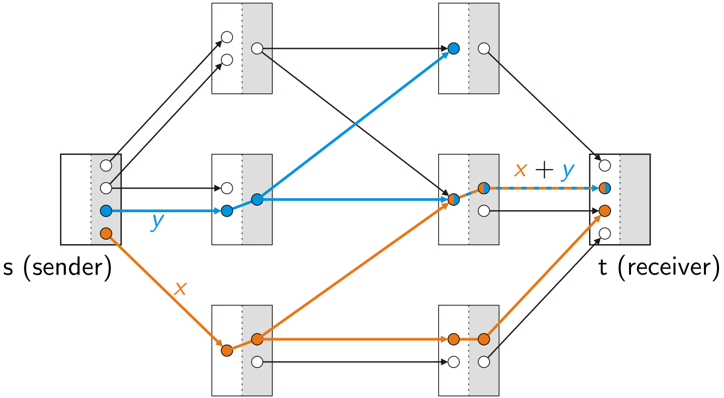

- Receiver gets signals $(x, x + y)$.
- Thanks to linear independence, received signals can be decoded to get original signals.

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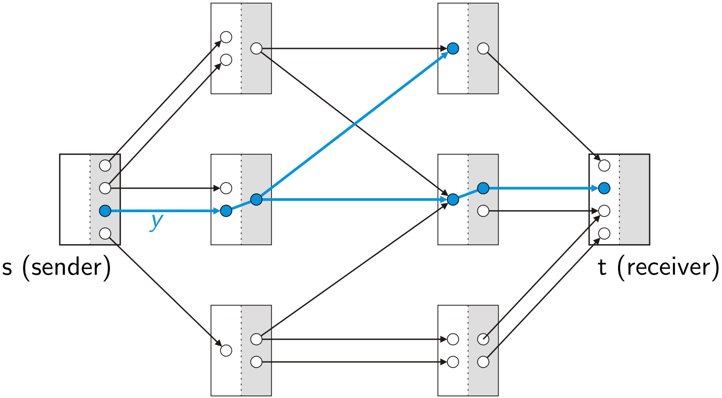

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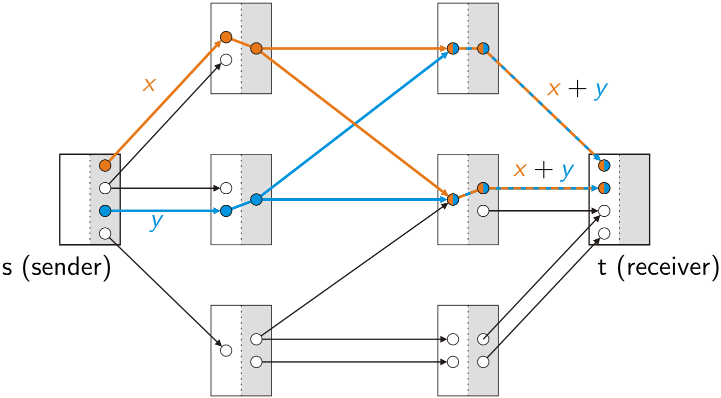

- Received signals are linearly dependent.

  → Receiver cannot properly decode.

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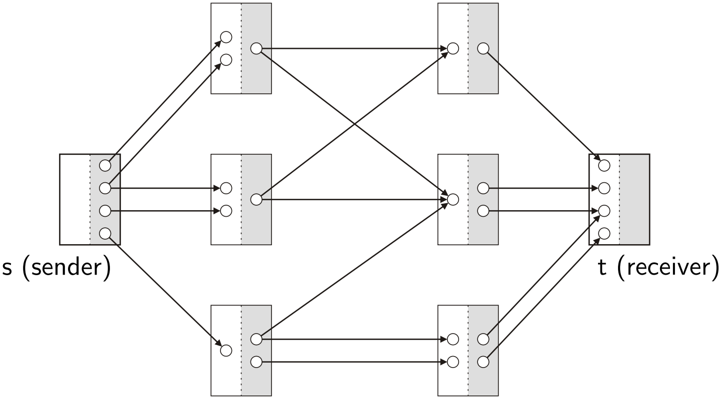
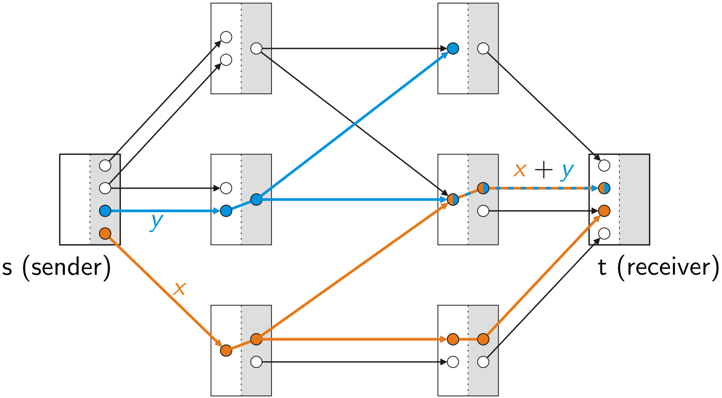


Goal

Route maximum number of decodable (i.e., linearly independent) signals from s to t.

Another representation of ADT flows



An ADT flow can be represented by set of used vertices.

- Concerning linear independence, exact wiring does not matter.
- Linear independence $\Leftrightarrow$ Adjacency matrix induced by used vertices in any two consecutive layers is full rank.

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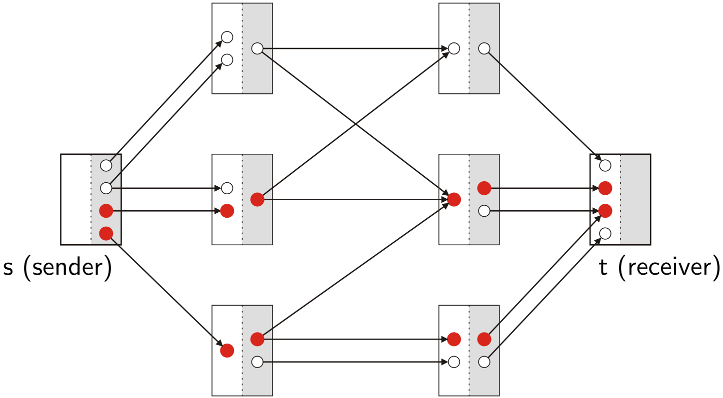

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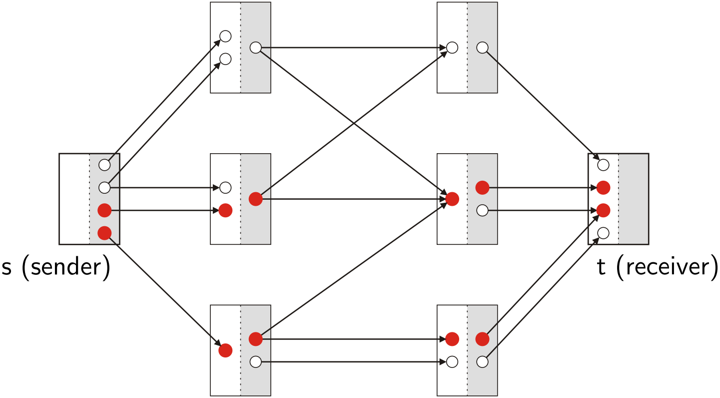

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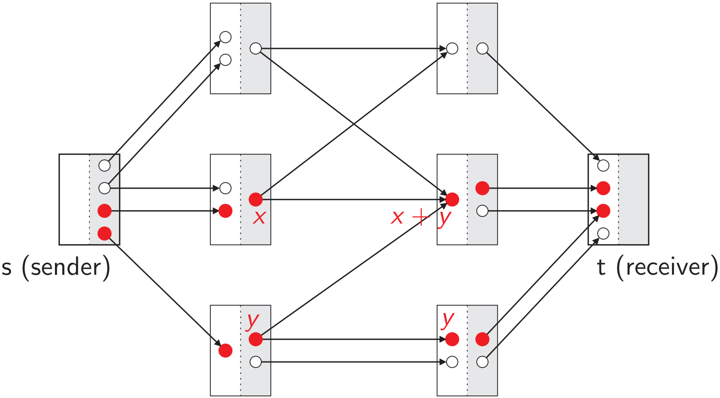

Propagation of signals from second to third layer:

$$(x, y) \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}_{\text{Induced adjacency matrix}} = (x + y, y).$$

Results on ADT network flows

Theorem ([Avestimehr, Diggavi, and Tse, 2007b])

A notion of cut was introduced such that:

$$\text{Max ADT flow} = \text{Min ADT cut}.$$

Theorem ([Amaudruz and Fragouli, 2009])

A maximum flow and a minimum cut can be found polynomial time.

In this talk: A more general flow model

- Max-flow min-cut theorem.
- Efficient optimization is possible (even with costs or capacities).
- Many other results can easily be deduced from matroid theory.
- Classical matroid algorithms can be used for optimization.
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2. **A flow model based on (poly-)linking systems**
   - Linking systems
   - Linking networks
   - Optimization in linking networks
   - Linking flow polytope
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Motivation of linking systems

Intuition

Relation between two finite sets V_1, V_2 that preserves matroid structure.

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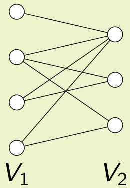
Induction of matroids (by a bipartite graph)

Let $G = (V_1 \cup V_2, E)$ be a bipartite graph and let $\mathcal{M} = (V_1, \mathcal{F})$ be a matroid.



$\{P_2 \subseteq V_2 \mid \exists P_1 \in \mathcal{F}$ such that $G[P_1 \cup P_2]$ contains a perfect matching$\}$ are independent sets of a matroid on V_2.

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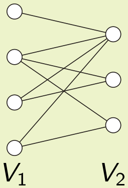

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$\longrightarrow$ Generalizations ?

Linking systems [Schrijver, 1978]

Definition: Linking system

A linking system between V_1 and V_2 is a triple (V_1, V_2, Λ) with $\emptyset \neq \Lambda \subseteq 2^{V_1} \times 2^{V_2}$ and satisfying:

i) $(P_1, P_2) \in \Lambda \Rightarrow |P_1| = |P_2|$,

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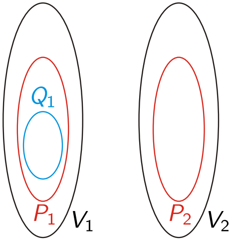

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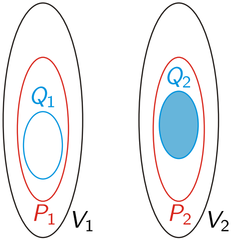

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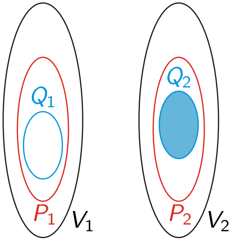

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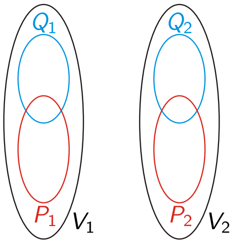

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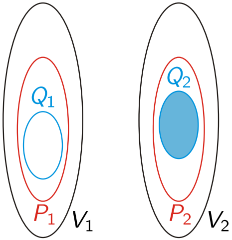

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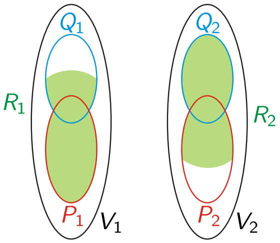

Linking systems: Examples I

Induced by bipartite graph

Let $G = (V_1 \cup V_2, E)$ be a bipartite graph. Then (V_1, V_2, Λ) is a linking system where

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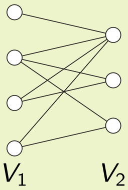

Linking systems: Examples II

Induced by matrix

Let $A \in \mathbb{R}^{n \times m}$ where V_1 resp. V_2 are the sets of row and column indices. Then (V_1, V_2, Λ) is a linking system where

$$\Lambda = \{(P_1, P_2) \in 2^{V_1} \times 2^{V_2} \mid A[P_1, P_2] \text{ is full rank}\}.$$

$$\begin{pmatrix} 1 & 2 & 5 & 0 & 10 \\ 0 & 0 & 3 & 3 & 7 \\ 0 & 1 & 2 & 1 & 4 \\ 2 & 0 & 7 & 2 & 8 \end{pmatrix}$$

Linking function (bisubmodular functions)

Definition of linking function

$$\lambda(P_1, P_2) = \max\{|Q_1| \mid (Q_1, Q_2) \in \Lambda, Q_1 \subseteq P_1, Q_2 \subseteq P_2\}.$$

Linking function determines linking system

$$(P_1, P_2) \in \Lambda \Leftrightarrow \lambda(P_1, P_2) = |P_1| = |P_2|.$$

Characterization of linking functions

i) $0 \leq \lambda(P_1, P_2) \leq \min\{|P_1|, |P_2|\}$,

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iii) $\lambda(P_1 \cap Q_1, P_2 \cup Q_2) + \lambda(P_1 \cup Q_1, P_2 \cap Q_2) \leq \lambda(P_1, P_2) + \lambda(Q_1, Q_2)$.

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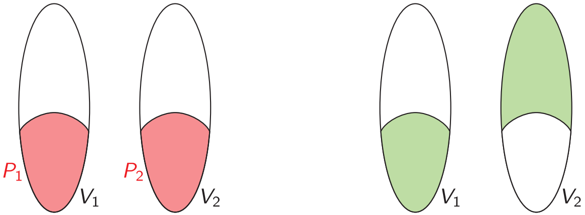
A matroidal property

Theorem
Let (V_1, V_2, Λ) *be a linking system.*

$$\mathcal{B}_\Lambda = \{P_1 \cup (V_2 \setminus P_2) \mid (P_1, P_2) \in \Lambda\}$$

forms the set of bases of a matroid. We denote this matroid by $M_\Lambda = (V_1 \cup V_2, \mathcal{F}_\Lambda)$.



The product of linking systems

linking system $\star$ linking system $\longrightarrow$ linking system.

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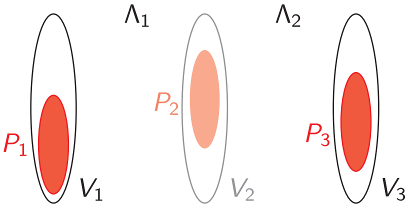
Linking system $\star$ linking system

Let $(V_1, V_2, \Lambda_1), (V_2, V_3, \Lambda_2)$ be two linking systems with linking functions λ_1, λ_2 and let

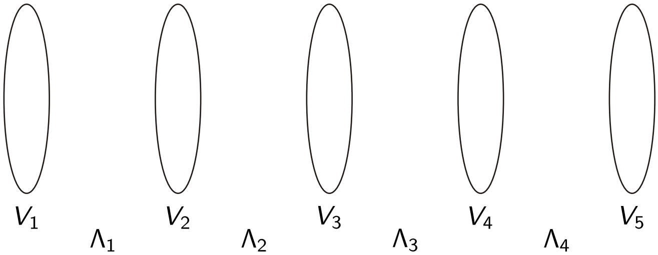
$$\Lambda_1 \star \Lambda_2 = \{(P_1, P_3) \in 2^{V_1} \times 2^{V_3} \mid \exists P_2 \subseteq V_2 \text{ with } (P_1, P_2) \in \Lambda_1, (P_2, P_3) \in \Lambda_2\}.$$

Then $(V_1, V_3, \Lambda_1 \star \Lambda_2)$ is a linking system with linking function

$$(\lambda_1 \star \lambda_2)(P_1, P_3) = \min_{P_2 \subseteq V_2} (\lambda_1(P_1, P_2) + \lambda_2(V_2 \setminus P_2, P_3)).$$



Linking networks (A flow model based on linking systems)



Definition: Linking network

Let $V_1, \ldots, V_r$ be finite disjoint sets and let $(V_i, V_{i+1}, \Lambda_i)$ be a linking system for $i \in \{1, \ldots, r-1\}$. Then $G = (V, \Lambda)$ is a linking network where $V = (V_1, \ldots, V_r)$, $\Lambda = (\Lambda_1, \ldots, \Lambda_{r-1})$.

Definition: Linking flow

Tuple $F = (F_1, \ldots, F_r)$ where $(F_i, F_{i+1}) \in \Lambda_i$ for $i \in \{1, \ldots, r-1\}$.

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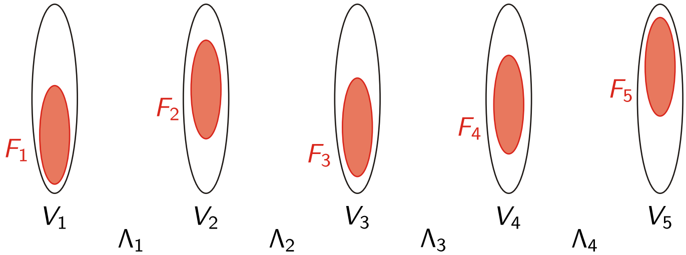

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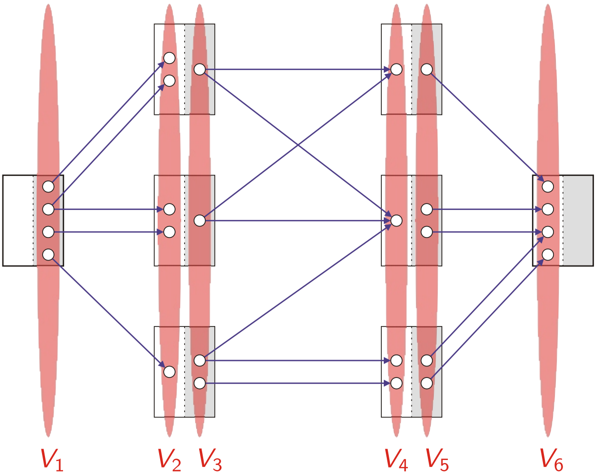

- In every node we add a complete bipartite graph.
- The linking systems alternate between:
  - Linking system induced by adjacency matrix.
  - Linking system induced by bipartite graph.

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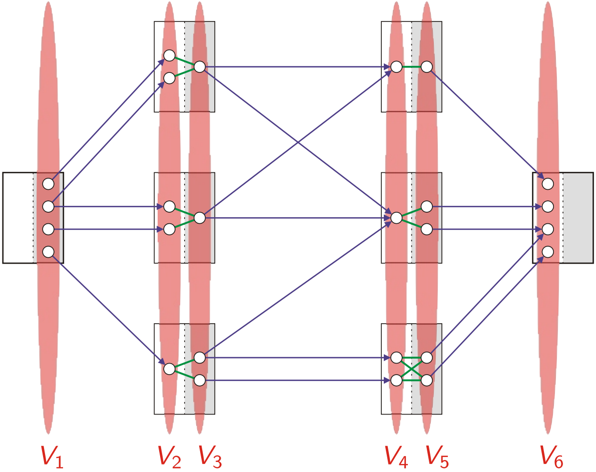

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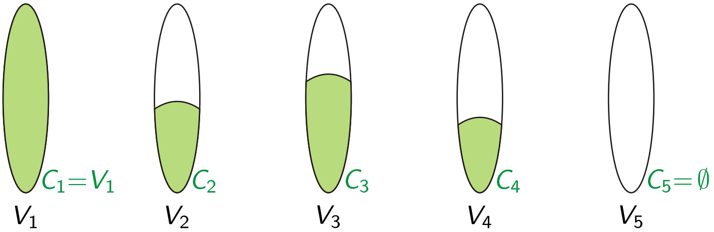
Source-destination cuts in linking networks



Definition: Cut

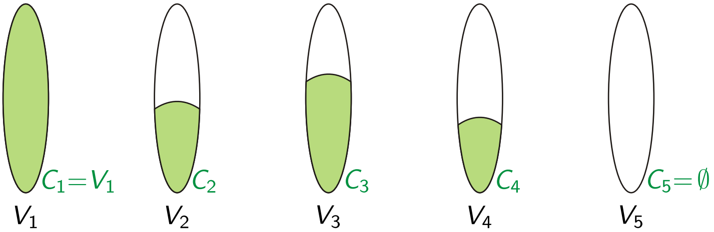
Tuple $C = (C_1, \ldots, C_r)$ with $C_i \subseteq V_i \; \forall i \in \{1, \ldots, r\}$, $C_1 = V_1$, $C_r = \emptyset$.

Definition: Value of a cut

$$\phi(C) = \sum_{i=1}^{r-1} \lambda_i(C_i, V_{i+1} \setminus C_{i+1}).$$

Min cut $\geq$ Max flow.

Source-destination cuts in linking networks



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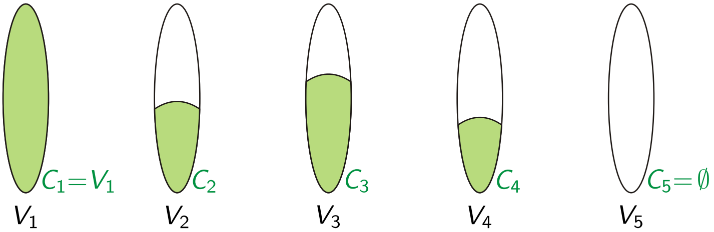

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Max-flow min-cut theorem in linking networks

Theorem: Max-flow min-cut

$$\text{Value of max-flow } = \text{Value of min-cut}$$

Proof.

- Let $\overline{\Lambda} = \Lambda_1 \star \cdots \star \Lambda_{r-1}$ with corresponding linking function $\overline{\lambda}$.
- Value of max flow $= \overline{\lambda}(V_1, V_r)$.
- Recall: Linking function of two chained linking systems $\Lambda_1 \star \Lambda_2$:

$$(\lambda_1 \star \lambda_2)(P_1, P_3) = \min_{P_2 \subseteq V_2} (\lambda_1(P_1, P_2) + \lambda_2(V_2 \setminus P_2, P_3)).$$

- By repeatedly applying the above formula we get

$$\overline{\lambda}(V_1, V_r) = \min \left\{ \phi(V_1 \cup \bigcup_{i=2}^{r-1} P_i) \mid P_2 \subseteq V_2, \ldots, P_{r-1} \subseteq V_{r-1} \right\}. \quad \square$$

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Some other properties

Submodularity of cut value

The value function of cuts $\phi(C) = \sum_{i=1}^{r-1} \lambda_i(C_i, V_{i+1} \setminus C_{i+1})$ is submodular.

Gammoid property

The set of attainable vertices in layer r (or any other fixed layer $l \in \{1, \ldots, r\}$) form a matroid, i.e,

$$\{F_r \mid (F_1, \ldots, F_r) \text{ linking flow}\}$$

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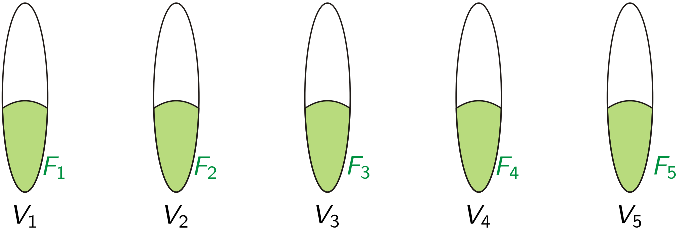
Finding flows through matroid union



Let $M_\Lambda = (\cup_{i=1}^{r} V_i, \mathcal{F}_\Lambda)$ be the union of the matroids $M_{\Lambda_1}, \ldots, M_{\Lambda_{r-1}}$.

- For any flow F,

$$F_1 \cup \left(\bigcup_{i=2}^{r-1} V_i\right) \cup (V_r \setminus F_r) \in \mathcal{F}_\Lambda.$$

- A maximum flow can be found by a matroid partitioning algorithm: Find a maximum independent set in M_Λ with as many elements in V_1 as possible (the number of elements in V_1 is the value of the flow).

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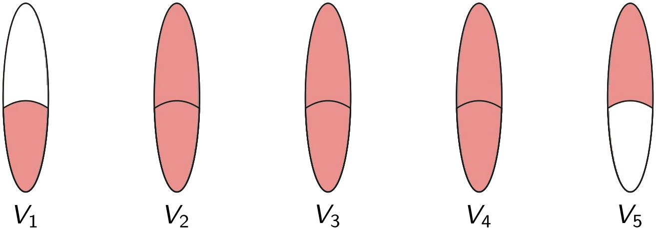

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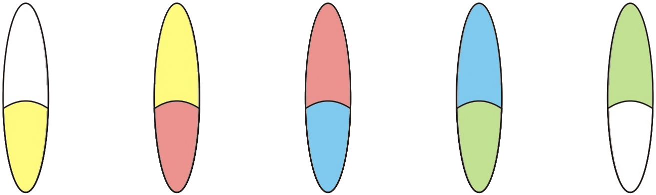

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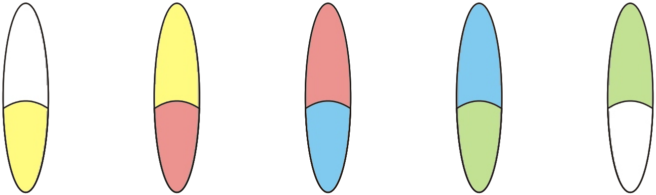

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Finding a minimum cut

- Let M_Λ^- be the matroid M_Λ restricted to $\cup_{i=1}^{r-1} V_i$.
- Let $\cup_{i=1}^{r-1} I_i$ be a maximum cardinality independent set M_Λ^- with $\cup_{i=2}^{r-1} V_i \subseteq \cup_{i=1}^{r-1} I_i$.
- By the Theorem of Nash-Williams we have

$$\underbrace{\rho_\Lambda^-(\cup_{i=1}^{r-1} V_i)}_{=|\cup_{i=1}^{r-1} I_i|} = \min_{A \subseteq \cup_{i=1}^{r-1} V_i} \left\{ |(\cup_{i=1}^{r-1} V_i) \setminus A| + \sum_{i=1}^{r-1} \rho_{\Lambda_i}(A) \right\}.$$

- Let A be a set attaining the above minimum (typically obtained as byproduct of a matroid partitioning algorithm).
- Expanding the minimum in the Nash-Williams formula, it can be shown that $(A \cap V_1, \ldots, A \cap V_{r-1}, V_r)$ is a minimum cut.

Linking flow polytope

Linking flow polytope

Let $G = (V, \Lambda)$ be a linking network. Its linking flow polytope is defined by

$$LFP(G) = \begin{cases} x(P_i) - x(V_{i+1} \setminus P_{i+1}) \leq \lambda_i(P_i, P_{i+1}) & \forall i \in \{1, \ldots, r-1\}, \\ & \forall P_i \subseteq V_i, \\ & \forall P_{i+1} \subseteq V_{i+1} \\ x(V_i) = x(V_{i+1}) & \forall i \in \{1, \ldots, r-1\} \\ x \in \mathbb{R}_+^{\sum_{i=1}^{r} |V_i|}. & \end{cases}$$

Theorem: Integrality of $LFP(G)$

$LFP(G)$ is integral and its vertices correspond to linking flows.

Integrality of LFP(G): Sketch of proof

$LFP(G)$ is a projection of the following polytope.

$$\begin{cases} x^i(P_i) - x^i(V_{i+1} \setminus P_{i+1}) \leq \lambda(P_i, P_{i+1}) & \forall i \in \{1, \dots, r-1\}, \\ & \forall P_i \subseteq V_i, P_{i+1} \subseteq V_{i+1} \\ x^i(V_i) = x^i(V_{i+1}) & \forall i \in \{1, \dots, r-1\} \\ x^i(v) = x^{i+1}(v) & \forall i \in \{1, \dots, r-1\}, \forall v \in V_{i+1} \\ x^i \in \mathbb{R}_+^{|V_i|} & \forall i \in \{1, \dots, r\} \end{cases}$$

- It suffices to show that the above polytope is integral.
- Choose a vertex of above polytope $\rightarrow$ defined by a set of equalities.
- We can uncross the equalities of this type for $i \in \{1, \dots, r-1\}$ such that if for a given i we have equalities for the tuples $(P_{i,1}, P_{i+1,1}), \dots, (P_{i,m}, P_{i+1,m})$ then the family

  $$\{P_{i,k} \cup (V_{i+1} \setminus P_{i+1,k}) \mid k \in \{1, \dots, m\}\}$$

  is laminar.
- Obtained equation system is totally unimodular.

Outline

1. Motivation (wireless information flow)
2. A flow model based on (poly-)linking systems
3. Conclusions

Conclusions and Outlook

- Linking networks: A flow model based on linking systems and generalizing the ADT model.
- Many nice properties:
  - Gammoid property.
  - Submodularity of cut-values.
  - Max-flow min-cut result.
- Efficient optimization is possible using standard matroid algorithms.
- Optimization with respect to costs is possible.
- Capacities can be incorporated by replacing linking systems with polylinking systems.

- Maybe one could go towards a more general model where the graph does not need to be acyclic.
- How to adapt current matroid algorithms to exploit special structure of linking systems?

Polylinking systems [Schrijver, 1978]

Definition: Polylinking system

A polylinking system between V_1 and V_2 is a triple (V_1, V_2, L) where $\emptyset \neq L \subseteq \mathbb{R}_+^{V_1} \times \mathbb{R}_+^{V_2}$ is a compact set satisfying:

i) $(x_1, x_2) \in L \Rightarrow |x_1| = |x_2|$,

ii) $(x_1, x_2) \in L, 0 \leq y_1 \leq x_1 \Rightarrow \exists y_2 \leq x_2$ with $(y_1, y_2) \in L$,

iii) $(x_1, x_2) \in L, 0 \leq y_2 \leq x_2 \Rightarrow \exists y_1 \leq x_1$ with $(y_1, y_2) \in L$,

iv) $(x_1, x_2), (y_1, y_2) \in L \Rightarrow \exists (z_1, z_2) \in L$ with $\begin{array}{l} x_1 \leq z_1 \leq x_1 \vee y_1, \\ y_2 \leq z_2 \leq x_2 \vee y_2. \end{array}$

References

A. Amaudruz and C. Fragouli. Combinatorial algorithms for wireless information flow. In *SODA '09: Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discrete Algorithms*, 2009.

A. S. Avestimehr, S. N. Diggavi, and D. N. C. Tse. A deterministic approach to wireless relay networks. In *Proceedings of Allerton Conference on Communication, Control, and Computing*, September 2007a. http://licos.epfl.ch/index.php?p=research_projWNC.

A. S. Avestimehr, S. N. Diggavi, and D. N. C. Tse. Wireless network information flow. In *Proceedings of Allerton Conference on Communication, Control, and Computing*, September 2007b. http://licos.epfl.ch/index.php?p=research_projWNC.

A. Schrijver. *Matroids and Linking Systems*. PhD thesis, Mathematisch Centrum, 1978.

A. Schrijver. Matroids and linking systems. *Journal on Combinatorial Theory, Series B*, 26(3):349–369, June 1979.