Dependent Randomized Rounding on the Spanning Tree Polytope with Applications in Multi-Objective Optimization

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Joint work with Chandra Chekuri and Jan Vondrák

Outline

1. **A short introduction to dependent rounding**
   - Motivation
   - Classical rounding approaches: a review by examples
   - Typical goals when designing rounding procedures

2. **Randomized swap rounding: rounding on matroid polytopes**
   - Motivating example
   - Randomized swap rounding on the spanning tree polytope
   - Example applications
   - Some short remarks about submodular functions

3. **Conclusion**

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   - Some short remarks about submodular functions

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Motivation

Many combinatorial problems become easy when relaxing integrality constraints.

How to profit from fractional solutions?

→ One promising approach: Rounding.

Useful problem properties for rounding procedures

- Small integrality gap
- "Weak constraints"
- Fractional solution has
  - few fractional values
  - fractional components generally have high values

→ Still for most problems it is not clear whether and how rounding procedures can be applied.

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An example for deterministic rounding

(Unweighted) Set Cover with bounded coverage

Given:
- A finite set U, collection of subsets $\mathcal{F} = \{S_1, \ldots, S_p\} \subseteq 2^U$.
- $|\{i \in [p] := \{1, \ldots, p\} \mid u \in S_i\}| \leq \ell$ for all $u \in U$.

Task: Find $I \subseteq [p]$ with $|I|$ minimum such that $\cup_{i \in I} S_i = U$.

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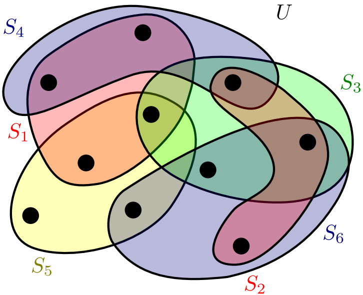

$\ell = 3$

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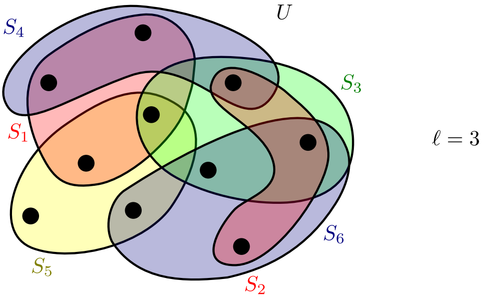

A ℓ-approximation can be obtained by rounding an LP solution.

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ILP
$$\begin{array}{llll} \min & \sum_{i \in [p]} x_i & & \\ & \sum_{i \in I : u \in S_i} x_i & \geq 1 & \forall u \in U \\ & x_i & \in \{0,1\} & \forall i \in [p] \end{array}$$

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Rounding step

If $x_i^* \geq 1/\ell$, set it to 1, else to 0 $\quad \Rightarrow \quad I = \{i \in [p] \mid x_i^* \geq 1/\ell\}$.

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Approximation quality and feasibility

i) Covering constraints are satisfied since for every inequality $\sum_{i \in I : u \in S_i} x_i^* \geq 1$ at least one x_i must be $\geq 1/\ell$.

ii) $|S| \leq \ell \cdot$ LP solution.

An example for randomized rounding (I)

Congestion minimization problem

Given: An undirected graph $G = (V, E)$, source destination pairs $(s_1, t_1), \dots, (s_p, t_p) \in V \times V$.

Task: For $i \in [p]$, find a s_i-t_i path $P^i \subseteq E$, such that the congestion $\max_{e \in E} |\{i \in [p] \mid e \in P^i\}|$ is minimized.

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\begin{aligned}
\min \quad & C & & \\
& f^i(\delta^+(v)) - f^i(\delta^-(v)) = \begin{cases} 1 & \text{if } v = s_i \\ -1 & \text{if } v = t_i \\ 0 & \text{otherwise} \end{cases} & & \forall i \in [p], v \in V \\
& \sum_{i=1}^{p} f^i(e) \leq C & & \forall e \in E \\
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A $O(\log m / \log \log m)$-approximation, where $m := |E|$, can be obtained by randomly rounding an optimal solution to the above LP.

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& f^i & \in [0,1]^E & \quad \forall i \in [p]
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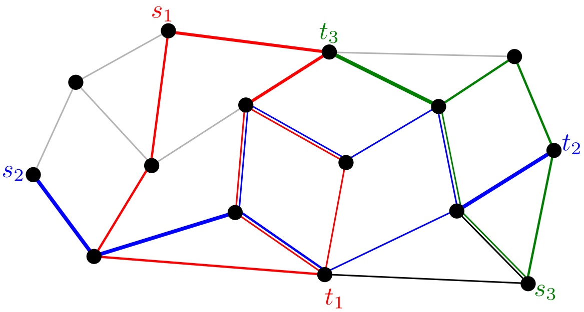
A $O(\log m / \log\log m)$-approximation

1. Determine an optimal solution $f^i \in [0,1]^E, i \in [p]$ to the LP.
2. Decompose each f^i into at most m paths $P^i_1, \dots P^i_{n_i}$ from s_i to t_i, i.e., $f^i = \sum_{k=1}^{n_i} \alpha^i_k \mathbf{1}_{P^i_k}$.
3. For each $i \in [p]$, choose randomly one path P^i out of $P^i_1, \dots, P^i_k$, where P^i_k is chosen with probability α^i_k.
4. Return $P^1, \dots, P^p$.



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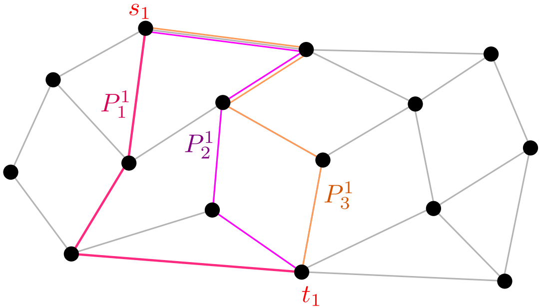

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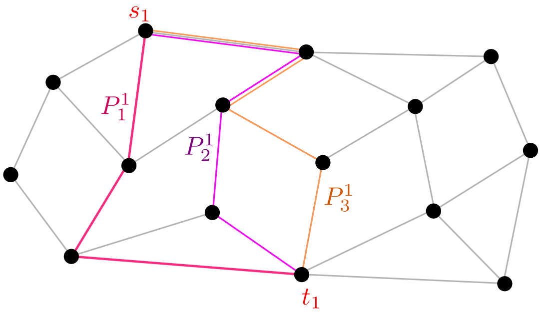

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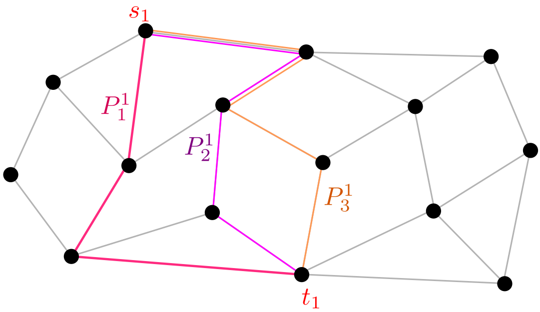

An example for randomized rounding (III)

Analysis of the algorithm

- Fix an edge e.
- $c^*(e) :=$ fractional congestion of e.
- $C(e) :=$ random congestion of e after rounding, i.e.,

$$C(e) = \sum_{i=1}^{p} \mathbf{1}_{e \in P^i}, \quad \text{where} \quad \mathbf{1}_{e \in P^i} = \begin{cases} 1 & \text{if } e \in P^i, \\ 0 & \text{otherwise.} \end{cases}$$

- The variables $\{\mathbf{1}_{e \in P^i} \mid i \in [p]\}$ are independent.
- $\mathbf{E}[C(e)] = c^*(e)$

Theorem (Chernoff bound)

Let $X_1, \ldots, X_n$ be independent random variables taking values in $[0,1]$. For $\mu \geq \mathbf{E}[\sum_{i=1}^{n} X_i]$ and $\delta > 0$ we have:

$$\Pr[\sum_{i=1}^{n} X_i \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}} \right)^{\mu}.$$

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- By union bound: $\Pr[\exists e \in E \text{ with } C(e) > (1+\delta)\mu(e)] \leq m \cdot \frac{1}{m^2} \leq 1/m$.
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Goals when designing rounding procedure?

- Maintain feasibility.
  → Can be obtained by careful dependent rounding.
- Rounded point should be "similar" to fractional one.
  → Typically obtained by independence of rounding and Chernoff bounds.

Main difficulty: these two goals are conflicting.

For example: how to round a point in

- the spanning tree polytope?
- a (poly-)matroid polytpe?
- the matching or b-matching polytope?
- the independent set polytope of the intersection of two (poly-)matroids?

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Multi-criteria spanning tree (I)

Terminology

- $G = (V, E)$: undirected graph.
- $\mathcal{T} \subseteq 2^E$: set of all spanning trees.
- $\mathbf{1}_U \in \{0,1\}^E$: incidence vector of the set $U \subseteq E$.
- $P_{ST} = conv(\{\mathbf{1}_T \mid T \in \mathcal{T}\})$: spanning tree polytope.

Multi-budgeted spanning tree problem

Given: G, a weight function $w : E \to [0,1]$, and p length functions $\ell_i : E \to [0,1]$ for $i \in [p]$ with corresponding budget $B_i \in \mathbb{Q}_+$.

Task: Find spanning tree of minimum weight respecting all budgets.

$$\min_{T \in \mathcal{T}} \; w(T) \quad \ell_i(T) \leq B_i \quad \forall i \in [p] \qquad \Leftrightarrow \qquad \min_{x \text{ vertex of } P_{ST}} \; w^T x \quad \ell_i^T x \leq B_i \quad \forall i \in [p]$$

These constraints are often weak, i.e., it is ok to violate them slightly.

$\to$ We then talk of multi-criteria optimization.

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- $P_{ST} = conv(\{\mathbf{1}_T \mid T \in \mathcal{T}\})$: spanning tree polytope.

Multi-budgeted spanning tree problem

Given: G, a weight function $w : E \to [0,1]$, and p length functions $\ell_i : E \to [0,1]$ for $i \in [p]$ with corresponding budget $B_i \in \mathbb{Q}_+$.

Task: Find spanning tree of minimum weight respecting all budgets.

$$\begin{array}{lll} \min\limits_{T \in \mathcal{T}} & w(T) & \\ & \ell_i(T) \le B_i & \forall i \in [p] \end{array} \quad \Leftrightarrow \quad \begin{array}{lll} \min\limits_{x \text{ vertex of } P_{ST}} & w^T x & \\ & \ell_i^T x \le B_i & \forall i \in [p] \end{array}$$

These constraints are often weak, i.e., it is ok to violate them slightly.

$\rightarrow$ We then talk of multi-criteria optimization.

Multi-criteria spanning tree (I)

Terminology

- $G = (V, E)$: undirected graph.
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Multi-criteria problem:

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$\xrightarrow{\text{LP relaxation}}$

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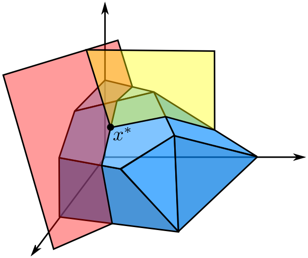

- The LP (with hard constraints) can be solved in polynomial time.
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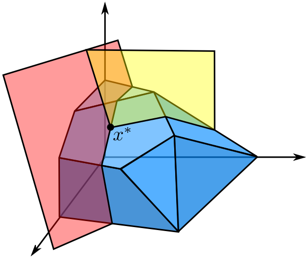

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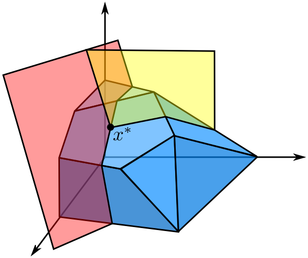

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Main framework

- Drop budget constraints and randomly round x^* to a vertex of P_{ST} close to x^*.

Problem: defining good rounding steps in P_{ST} is not easy.
Idea: profit from combinatorial knowledge about spanning trees.

- We work on a convex decomposition of the point $x^* \in P$ to round: $x^* = \sum_{i=1}^{m} \beta_i \mathbf{1}_{T_i}$, where $T_1, \ldots, T_m$ are spanning trees.
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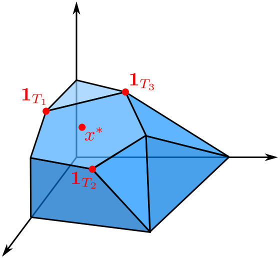

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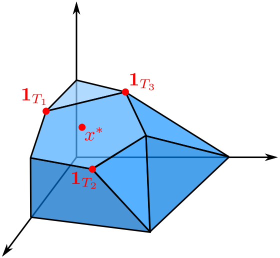

Rounding using the convex decomposition

- We start with $x_1 := x^*$ and iteratively reduce the number of terms in the convex representation using a Merge operation.

$$
\begin{aligned}
x_1 &= \underbrace{\beta_1 \mathbf{1}_{T_1} + \beta_2 \mathbf{1}_{T_2}} + \beta_3 \mathbf{1}_{T_3} + \dots \beta_m \mathbf{1}_{T_m} \\
x_2 &= \underbrace{(\beta_1 + \beta_2)\mathbf{1}_{T_{1:2}} + \beta_3 \mathbf{1}_{T_3}} + \dots \beta_m \mathbf{1}_{T_m} \quad & \mid T_{1:2} = Merge(\beta_1, T_1, \beta_2, T_2) \\
x_3 &= (\beta_1 + \beta_2 + \beta_3)\mathbf{1}_{T_{1:3}} + \dots \beta_m \mathbf{1}_{T_m} \quad & \mid T_{1:3} = Merge(\beta_1 + \beta_2, T_{1:2}, \beta_3, T_3) \\
&\vdots \\
x_k &= (\textstyle\sum_{i=1}^{k} \beta_i)\mathbf{1}_{T_{1:k}} + \textstyle\sum_{i=k+1}^{m} \beta_i \mathbf{1}_{T_i} \quad & \mid T_{1:k} = Merge(\textstyle\sum_{i=1}^{k-1} \beta_i, T_{1:k-1}, \beta_k, T_k) \\
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Merging two spanning trees by swaps

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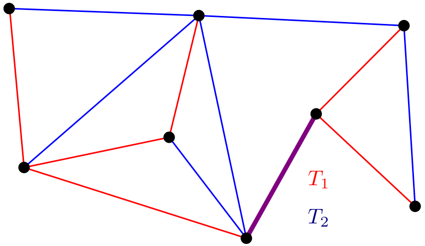

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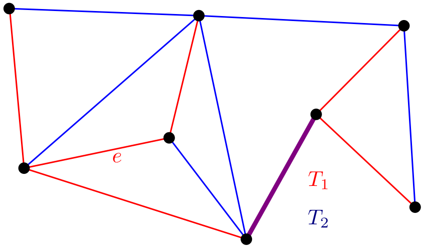

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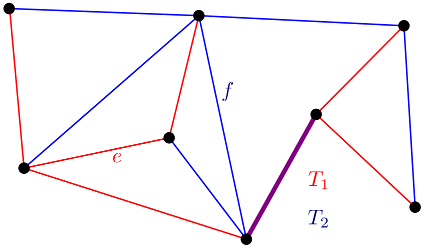

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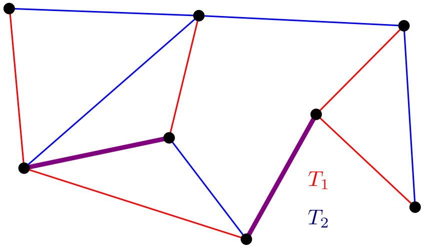

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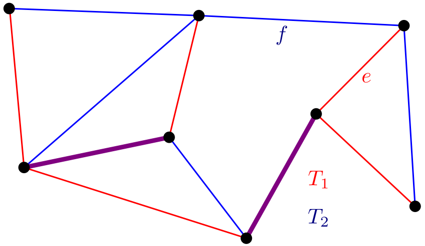

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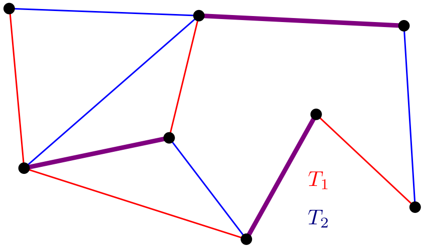

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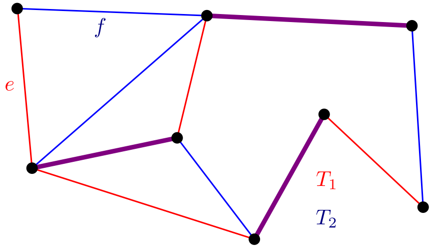

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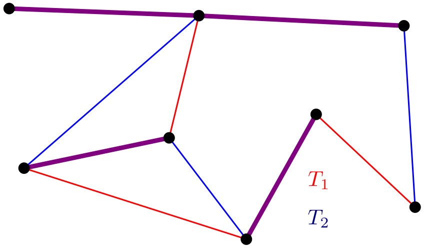

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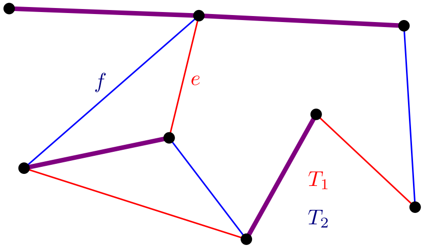

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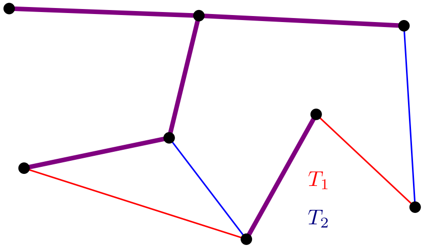

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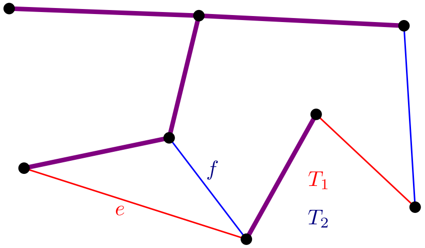

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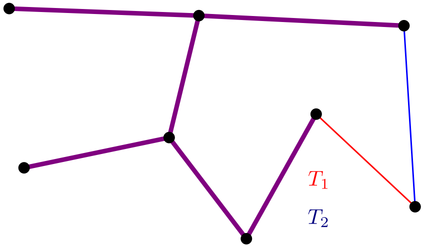

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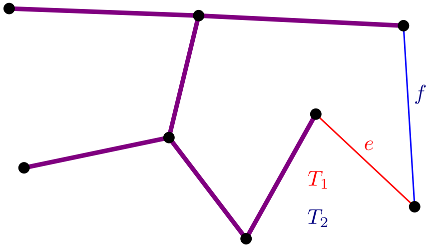

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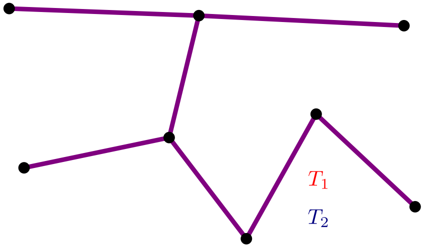

Concentration bounds (I)

- Assume for simplicity $x^* = \beta_1 \mathbf{1}_{T_1} + \beta_2 \mathbf{1}_{T_2}$.

Properties of the Merge procedure

Let y^1 be the point obtained after applying one merge operation to x^*.

i) $\mathbf{E}[y^1] = x^*$.

ii) Exactly two components change and their sum remains constant.

Theorem

From the above properties the following negative correlation property of a rounded tree T can be derived. For any $U \subseteq E$, we have

- $\Pr[U \subseteq T] \leq \prod_{e \in U} x^*(e)$,
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Such negative correlation is sufficient to get Chernoff bounds (Panconesi and Srinivasan [1997]).

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Let $\ell : E \to [0,1]$ and let $\ell(x) := \sum_{e \in E} x_e \ell(e)$ for $x \in [0,1]^E$.

Theorem (Chernoff bounds for swap rounding)

Let T be a random tree obtained by randomized swap rounding. For $\mu \geq \ell(x^*)$ and $\delta > 0$ we have:

$$\Pr[\ell(T) \geq (1+\delta)\mu] \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu .$$

- Similar bounds can be shown for the lower tail (and even for submodular functions).

Linear functions do not change much through randomized swap rounding!

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Concentration bounds (II)

Let $\ell : E \to [0,1]$ and let $\ell(x) := \sum_{e \in E} x_e \ell(e)$ for $x \in [0,1]^E$.

Theorem (Chernoff bounds for swap rounding)

Let T be a random tree obtained by randomized swap rounding. For $\mu \geq \ell(x^*)$ and $\delta > 0$ we have:

$$\Pr[\ell(T) \geq (1+\delta)\mu] \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu .$$

- Similar bounds can be shown for the lower tail (and even for submodular functions).

Linear functions do not change much through randomized swap rounding!

Application to multi-criteria spanning tree

$$\begin{aligned} \min_{x \in P_{ST}} \quad & w^T x \\ & \ell_i^T x \preceq B_i \quad \forall i \in [p] \end{aligned}$$

$\rightarrow$ Again, let x^* be an optimal solution to this LP.

Theorem

Let $\epsilon \in [0,1]$. It suffices to consider $O(1/\epsilon)$ independent outputs of the swap rounding algorithm, to obtain with high probability at least one spanning tree T that is a $(1+\epsilon, O(\log p / \log\log p))$-approximation for x^*, i.e., there is a constant $c(\epsilon)$ such that

$$\begin{aligned} w(T) &\leq (1+\epsilon) w(x^*) \\ \ell_i(T) &\leq \frac{c(\epsilon)\log p}{\log\log p} \ell_i(x^*) \quad \forall i \in [p] \end{aligned}$$

Remarks

- Since we have a $(1+\epsilon, O(\log p / \log\log p))$-approximation with respect to x^*, we have the same guarantee with respect to OPT.
- The above result holds even if p is not constant.
- If p is constant, a polynomial $(1+\epsilon, 1+\epsilon)$-approximation can be obtained by a preliminary guessing step.

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Proof of $(1+\epsilon, 4\log p/\log\log p)$-approximation

- $T :=$ spanning tree obtained from x^* by randomized swap rounding.
- $1+\delta := 4\log p/\log\log p$ and we assume for simplicity $\epsilon \geq 4/p$.

Probability that budget i is violated:
Using Chernoff bound for swap rounding:

$$\Pr[\ell_i(T) > (1+\delta)B_i] \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^{B_i} \overset{\boxed{B_i \geq 1}}{\leq} \frac{e^\delta}{(1+\delta)^{1+\delta}} \leq \left(\frac{e}{1+\delta}\right)^{1+\delta}$$

$$= \left(\frac{e\log\log p}{4\log p}\right)^{\frac{4\log p}{\log\log p}} \leq \left(\frac{1}{\sqrt{\log p}}\right)^{\frac{4\log p}{\log\log p}} = \frac{1}{p^2}.$$

Probability that some budget is violated:
Using the union bound:

$$\Pr[\ell_i(T) > (1+\delta)\ell_i(x^*) \text{ for some } i] \leq p \cdot \frac{1}{p^2} = \frac{1}{p} \leq \frac{\epsilon}{4}.$$

Probability that objective function is not fine:
Using Markov's inequality:

$$\Pr[w(T) > (1+\epsilon)w(x^*)] \leq \frac{\mathbf{E}[w(T)]}{(1+\epsilon)w(x^*)} \overset{\boxed{\mathbf{E}[w(T)] = w(x^*)}}{=} \frac{1}{1+\epsilon} \leq 1 - \frac{\epsilon}{2}.$$

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Probability that objective function or some budget is not ok:
Using union bound on the two probabilities above:

$$\Pr[T \text{ not fine}] \leq 1 - \frac{\epsilon}{2} + \frac{\epsilon}{4} = 1 - \frac{\epsilon}{4}.$$

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A short comment about submodular functions

Definition
Let S be a finite set. A function $f : 2^S \to \mathbb{R}$ is submodular if

$$f(A \cup \{s\}) - f(A) \geq f(B \cup \{s\}) - f(B) \quad \forall A \subseteq B \subseteq S, s \in S.$$

- Submodular functions are interesting candidates for utility functions since they can model diminishing returns.
- Since submodular functions are only defined on a discrete set, an extension is typically used in LP relaxations. A useful candidate is the multilinear extension F defined for $x \in [0,1]^E$ by

$$F(x) = \sum_{R \subseteq S} \left(\prod_{i \in R} x_i \right) \left(\prod_{i \notin R} (1 - x_i) \right).$$

- The upper tail concentration bounds of randomized swap rounding also hold for the above submodular extension.

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Outline

Conclusions

- Randomized rounding is a powerful tool in many settings.
- Randomized swap rounding allows to profit from combinatorial knowledge of the underlying problem by
  - **i)** Representing the point to round as a convex combination of vertices of the underlying polytope.
  - **ii)** Applying merging steps on the terms in the convex combination.
- The spanning tree polytope (or more generally matroid polytopes) have nice combinatorial properties for applying rounding procedures.

- In which other settings is the general swap rounding framework useful? (So far, we have some results on matroid intersection and b-matchings)
- Can the approach be derandomized in some non-trivial settings?

References

A. Panconesi and A. Srinivasan. Randomized distributed edge coloring via an extension of the chernoff–hoeffding bounds. *SIAM Journal on Computing*, 26(2):350–368, 1997. ISSN 0097-5397. doi: http://dx.doi.org/10.1137/S0097539793250767.